Note: In this HW, the base field will always be $\mathbb{R}$. An r-plane in $\mathbb{P}^n$ is an $(r+1)$-dimensional subspace of $\mathbb{R}^{n+1}$, i.e., $\mathbb{G}_r\mathbb{P}^n = G(r+1, n+1)$. Take Plücker coordinates for $G(r,n)$ using lexicographic ordering on the column indices $j\colon 1 \le j_1 < \cdots < j_r \le n$. (So $j < j'$ if first nonzero entry of $j - j'$ is negative.) Also, we embed $\mathbb{R}^n \subset \mathbb{P}^n$ by sending $(x_1, \ldots, x_n)$ to the point with homogeneous coordinates $(x_1, \ldots, x_n, 1)$, and we embed $G(2,4) = \mathbb{G}_1\mathbb{P}^3 \subset \mathbb{P}^5$ via the Plücker embedding.

Problem 1.

(a) Find the coordinates of

$$L = \begin{pmatrix} 1 & 3 & 0 & -1 \\ 2 & 7 & 1 & 3 \end{pmatrix} \in G(2,4)$$

with respect to the chart $(U_{1,2}, \phi_{1,2})$ and with respect to the chart $(U_{1,3}, \phi_{1,3})$. Assume we are working over the reals. So the coordinates will be in $\mathbb{R}^m$ for some m.

(b) Find the transition function from $U_{1,2}$ to $U_{1,3}$ for $G(2,4)$ (as a mapping $\mathbb{R}^m \to \mathbb{R}^m$ for some m).

(c) Use (b) to verify (a).

Problem 2. A *plane* in $\mathbb{R}^n$ is a set of the form $L = p + W$ where $p \in \mathbb{R}^n$ and W is a 2-dimensional subspace of $\mathbb{R}^n$. From linear algebra, we know $L = W$ if and only if $0 \in L$.

(a) Let L and M be planes in $\mathbb{R}^3$. If L and M are parallel, then $L \cap M = \emptyset$, and thus $\dim(L \cap M) = -1$. It is also possible that L and M meet in a line, and thus $\dim(L \cap M) = 1$. If $L = M$, then $\dim(L \cap M) = 2$. Show that it is impossible for $\dim(L \cap M) = 0$, i.e., for L and M to meet in a point. (An inequality from class may help.)

(b) Give an example of two planes in $\mathbb{R}^4$ meeting in a single point.

Problem 3. For each of the following, determine whether the given point $p \in \mathbb{P}^5$ is the set of Plücker coordinates for a line in $\mathbb{P}^3$, and, if so, find homogeneous coordinates for the corresponding line in $\mathbb{P}^3$ (i.e., element of $G(2,4) = \mathbb{G}_1\mathbb{P}^3$):

(a) $p = (2, 0, -1, 0, 1, 1)$

(b) $p = (1, 3, 2, 0, 1, 3)$.

Problem 4. Consider the line in $\mathbb{R}^4$ with parametric equation

$$p(s) = (1, 2, 3, 4) + s(1, 1, 0, 0).$$

Embed $\mathbb{R}^4$ in $\mathbb{P}^4$ via $(x_1, \ldots, x_4) \mapsto (x_1, \ldots, x_4, 1)$. Let L be the closure of the image of $p(s)$ in $\mathbb{P}^4$ to get a line in $\mathbb{P}^4$. (Remember: a line is determined by two points.) Find the Plücker coordinates of L.

Problem 5. We would like to show that the set of all lines in three-space that lie in a given plane and pass through a given point forms a line L such that $L \subset \mathbb{G}_1\mathbb{P}^3 \subset \mathbb{P}^5$.

(a) Let $p, q \in \mathbb{P}^3$ and let P, Q be planes in $\mathbb{P}^3$ (i.e., 3-dimensional subspaces of $\mathbb{R}^4$). Suppose $p \in P$ and $q \in Q$. Prove there is a linear isomorphism $\mathbb{R}^4 \to \mathbb{R}^4$ that, when considered as a mapping $\mathbb{P}^3 \to \mathbb{P}^3$, sends p to q and P to Q.

(b) Let $p \in \mathbb{P}^3$ and let P be a plane in $\mathbb{P}^3$. Let L be the set of all lines in $\mathbb{P}^3$ that lie in P and pass through the point p. Since L consists of lines in $\mathbb{P}^3$, we have $L \subset \mathbb{G}_1\mathbb{P}^3$. Show that L is a line in $\mathbb{P}^5$. (Hint: the previous part of this problem can simplify matters. Find the Plücker coordinates of any line meeting the given conditions.)