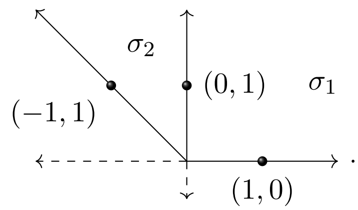
Math 341 for Wednesday, Week 12

PROBLEM 1. Consider the cone $\sigma = \mathbb{R}_{\geq 0}(0,1) + \mathbb{R}_{\geq 0}(3,-1)$. Describe the affine toric variety U_σ. (There are four generators and three relations for the semigroup S_σ. The three relations do not involve coefficients higher than 2.)

PROBLEM 2. Let Δ be the fan with two maximal cones $\sigma_1 = \mathbb{R}_{\geq 0}(1,0) + \mathbb{R}_{\geq 0}(0,1)$ and $\sigma_2 = \mathbb{R}_{\geq 0}(-1,1) + \mathbb{R}_{\geq 0}(0,1)$:



(a) Restricting to the reals, have $U_{\sigma_1} = \mathbb{R}^2$ with natural coordinates u and v corresponding to e_1 and e_2, respectively. Describe the toric variety $X(\Delta)$ by identifying U_{σ_2} and the gluing mapping $U_{\sigma_1} \dashrightarrow U_{\sigma_2}$ (the dashed arrow means that the mapping is not defined at all points of U_{σ_1}). Try to order the coordinates so that the gluing gives a $\mathbb{P}^1$ in the first coordinates.

(b) There is a mapping of lattices

$$N = \mathbb{Z}^2 \to \mathbb{Z} = N'$$
$$(u,v) \mapsto u$$

which naturally maps Δ to a fan Δ' having maximal cones $\tau_1 = \mathbb{R} \cdot 1$ and $\tau_2 = \mathbb{R} \cdot (-1)$. For $i = 1, 2$, the image of σ_i under the mapping is τ_i and induces a mapping $U_{\sigma_i} \to U_{\tau_i}$. These two maps glue together to give a mapping

$$\pi \colon X(\Delta) \to X(\Delta') = \mathbb{P}^1.$$

Describe this construction by filling in the blanks below in terms of the coordinates (u,v) on U_{σ_1}:

$$\begin{array}{ccccc} U_{\sigma_2} \cong \mathbb{R}^2 & \quad & (-,-) \dashleftarrow\dashrightarrow (u,v) & \quad & \mathbb{R}^2 \cong U_{\sigma_1} \\ \downarrow & & \downarrow \qquad\quad\;\; \downarrow & & \downarrow \\ U_{\tau_2} \cong \mathbb{R} & & - \dashleftarrow\dashrightarrow - & & \mathbb{R} \cong U_{\tau_1}. \end{array}$$

(c) Over the reals, the space $X(\Delta)$ is found by gluing two copies of $\mathbb{R}^2$. Can you identify this space and the mapping π? (Some things that might help: Think of $\mathbb{P}^1$ as S^1. What are the fibers of the mapping $X(\Delta) \to \mathbb{P}^1$?)

PROBLEM 3. Let P be the hexagon formed as the convex hull as $(0,0)$, $(1,0)$, $(0,1)$, $(2,1)$, $(1,2)$, and $(2,2)$.

(a) Draw the fan Δ_P for the corresponding toric variety, labeling the first lattice points along its 1-dimensional cones.

(b) Choose two adjacent 2-dimensional cones in Δ_P, construct the two corresponding affine toric varieties, and show the gluing instructions.

(c) Show that $X(\Delta_P)$ smooth.
(d) Calculate the Chow ring, $A^{\bullet}X(\Delta_P)$.
(e) Calculate the cohomology $H^k X(\Delta_P)$ for all k.