Math 341 for Friday, Week 11

Problem 1. We have seen that $H^k S^1 \approx \mathbb{R}$ for $k = 0, 1$ and zero otherwise. Since S^1 is a deformation retraction of the punctured plane, they have isomorphic cohomology groups. Use Mayer-Vietoris to compute the cohomology of the plane with n punctures. To skip some technical details, let $p_i = (i, 0)$ for $i = 1, \ldots, n$, and let $M_n := \mathbb{R}^2 \setminus \{p_1, \ldots, p_n\}$. The problem is to compute the cohomology of M_n for $n \geq 1$. (Explicit deformations are not required.)

Problem 2. In the following, feel free to identify functions on S^1 with functions on $\mathbb{R}$ having period 2π.

(a) Suppose f is a continuous function on $\mathbb{R}$ and $\int_0^{2\pi} f(t)\,dt = 0$. Show there is a periodic function g on $\mathbb{R}$ such that $g'(x) = f(x)$ for all $x \in [0, 2\pi]$.
(b) Let $\iota\colon S^1 \to \mathbb{R}^2$ be the inclusion mapping, and define $\omega = \iota^*(-y\,dx + x\,dy) \in \Omega^1 S^1$. Compute $\int_{S^1} \omega$.
(c) Explain why

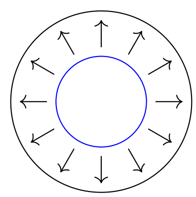
$$\begin{aligned} \phi\colon H^1 S^1 &\to \mathbb{R} \\ [\eta] &\mapsto \int_{S^1} \eta \end{aligned}$$

is well-defined.
(d) Prove that $H^1 S^1 = \mathbb{R}$ by showing ϕ is a bijection.

(In fact, this method extends to show that $H^n M = \mathbb{R}$ for any compact, connected, orientable, n-manifold M. The top cohomology is spanned by any "orientation form". If M is compact, connected, and non-orientable, then $H^n M = 0$.)

Problem 3. Let $S^n := \left\{x \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\right\}$. Let $N = (0, \ldots, 0, 1)$ and $S = (0, \ldots, 0, -1)$ be the "north and south poles" of S^n. Define $U := S^n \setminus \{N\}$ and $V := S^n \setminus \{S\}$. Then projection from the poles give diffeomorphisms of U and of V with $\mathbb{R}^n$. Hence, both U and V are contractible.

(a) Find an explicit diffeomorphism from $U \cap V$ to $S^{n-1} \times (-1, 1)$, a cylinder over S^{n-1}. (Hint: slide points out without changing their heights.) The picture below shows a slice of S^n in blue at height $x_{n+1} =$ constant and an indication of the diffeomorphism:



$r = 1$.

(b) What is the cohomology of S^0?
(c) Use Mayer-Vietoris to compute the cohomology of S^n for $n \geq 1$:

$$H^k S^n = \begin{cases} \mathbb{R} & \text{if } k = 0, n \\ 0 & \text{otherwise,} \end{cases}$$

(Consider the cases $k = 1$ and $k > 1$ separately.)