PROBLEM 1. Find a flaw in the following argument. Let M be an oriented $n$manifold without boundary. Then $H^nM \neq 0$. To prove this, let $\omega \in \Omega^n M$ be an n-form with compact support such that $\int_M \omega \neq 0$. Such forms always exist. Then ω is not a coboundary, i.e., ω is not exact. For otherwise, there would exist $\eta \in \Omega^{n-1}M$ such that $d\eta = \omega$, and then Stokes' theorem coupled with the fact that $\partial M = \emptyset$ would imply

$$\int_M \omega = \int_M d\eta = \int_{\partial M} \eta = 0.$$

(Compare the above to a similar argument given in Proposition 11.9.)

PROBLEM 2. Let M and N be smooth manifolds. Show that homotopy, $\sim$, is an equivalence relation on the set of mappings $f\colon M \to N$. As explained in the text (Remark 11.22), it is enough to establish this result for continuous homotopies.

PROBLEM 3. A *vector field* on $\mathbb{R}^n$ is a function $F\colon \mathbb{R}^n \to \mathbb{R}^n$. (Imagine attaching the vector $F(p)$ to p for each point $p \in \mathbb{R}^n$.) The *flow form* for $F = (F_1, \dots, F_n)$ is

$$\omega_F = \sum_{i=1}^n F_i\, dx_i \in \Omega^1\mathbb{R}^n.$$

If $I \subset \mathbb{R}$ is an interval and $\gamma\colon I \to \mathbb{R}^n$ is a parametrized curve in $\mathbb{R}^n$, then

$$\int_\gamma F \cdot \vec{t} := \int_\gamma \omega_F := \int_I \gamma^* \omega_F$$

measures how much the vector field is flowing along γ. The *flux form* for F is

$$\omega^F := \sum_{i=1}^n (-1)^{i-1} F_i\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n.$$

Let $D \subset \mathbb{R}^{n-1}$ be a rectangle (or any "reasonable" domain in $\mathbb{R}^n$), and let $S\colon D \to \mathbb{R}^n$ be a parametrized hypersurface in $\mathbb{R}^n$. Then

$$\int_S F \cdot \vec{n} := \int_S \omega^F := \int_D S^* \omega^F$$

measures the *flux* of F through the hypersurface S (i.e., how much does F tend to flow from one side of the hypersurface to the other). It "adds up" the components of F normal to hypersurface.

If $\phi\colon \mathbb{R}^n \to \mathbb{R}$, then the *gradient* of ϕ is the vector field

$$\nabla\phi := \left(\frac{\partial\phi}{\partial x_1}, \dots, \frac{\partial\phi}{\partial x_n}\right) : \mathbb{R}^n \to \mathbb{R}^n.$$

If $F = \nabla\phi$, then ϕ is called a *potential* for F. If $n = 3$, the *curl* of F is the vector field

$$\operatorname{curl} F := \left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}, \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}, \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right).$$

If $u \in \mathbb{R}^n$ is a unit vector, then the dot product $u \cdot \operatorname{curl} F$ measures how much F is circulating about u. Finally, again in the case $n = 3$, the *divergence* of F is the $\mathbb{R}$-valued function

$$\operatorname{div} F := \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}.$$

The quantity $\operatorname{div}(F)(p)$ measures how much F is diverging from the point p.

Together, Stokes' theorem and the fact that

$$H^k\mathbb{R}^n = \begin{cases} \mathbb{R} & \text{if } k = 0, \\ 0 & \text{if } k > 0 \end{cases}$$

sums up a lot of classical vector calculus.

(a) Justify the following notation for the de Rham complex for $\mathbb{R}^3$:

$$0 \to \Omega^0\mathbb{R}^3 \xrightarrow{\nabla} \Omega^1\mathbb{R}^3 \xrightarrow{\text{curl}} \Omega^2\mathbb{R}^3 \xrightarrow{\text{div}} \Omega^3\mathbb{R}^3 \to 0.$$

(Note: you will want to take the basis $dy \wedge dz, -dx \wedge dz, dx \wedge dy$ for $\Omega^2\mathbb{R}^3$).

(b) What does the fact that $H^0\mathbb{R}^n = \mathbb{R}$ say about functions $f\colon \mathbb{R}^n \to \mathbb{R}$? State your result in terms of elementary vector calculus, without reference to differential forms, exterior derivatives, etc.

(c) Prove the conclusion of the previous result directly, againg using elementary vector calculus. (Hint: for each point $p \in \mathbb{R}^n$, let $\gamma_p\colon [0,1] \to \mathbb{R}^n$ be a curve from 0 to p. Use the function $f \circ \gamma_p$ and one-variable calculus.)

(d) Suppose a vector field F in $\mathbb{R}^n$ has a potential ϕ. Using results from our course, prove that the flow of F along any curve $\gamma\colon [0,1] \to \mathbb{R}^n$ only depends on the endpoints of γ. (Such a vector field is called *conservative*.)

(e) What does the fact that $H^1\mathbb{R}^3 = 0$ say in the classical language described above?

(f) Same question for $H^2\mathbb{R}^3$.