Problem 1. Let V and W be vector spaces over an arbitrary field $\Bbbk$ and suppose that V has finite dimension n (no restriction on the dimension of W). In both of the following problems, it is useful to fix a basis $e_1, \dots, e_n$ for V.

(a) Show that

$$\mathrm{Hom}(V, W) \approx W^n = \underbrace{W \times \cdots \times W}_{n \text{ factors}}.$$

(b) There is a mapping of vector spaces

$$\begin{aligned} \alpha \colon V^* \otimes W &\to \mathrm{Hom}(V, W) \\ \phi \otimes w &\mapsto [v \mapsto \phi(v)w]. \end{aligned}$$

It's induced by the corresponding bilinear mapping from $V^* \times W$ via the universal property for tensor products. Show that this mapping is an isomorphism by finding (with proof) its inverse $\beta \colon \mathrm{Hom}(V, W) \to V^* \otimes W$. (Use your basis for V and the dual basis for V^*.)

Problem 2. The cross product

$$\begin{aligned} \mathbb{R}^3 \times \mathbb{R}^3 &\to \mathbb{R}^3 \\ (u, v) &\mapsto u \times v \end{aligned}$$

is an alternating, multilinear mapping. Hence, it factors through a unique mapping $\Lambda^2\mathbb{R}^3 \to \mathbb{R}^3$, i.e., to an element of $\mathrm{Hom}(\Lambda^2\mathbb{R}^3, \mathbb{R}^3)$. By the previous exercise, we can identify this element with an element of $(\Lambda^2\mathbb{R}^3)^* \otimes \mathbb{R}^3$, and hence with an element of $\Lambda^2((\mathbb{R}^3)^*) \otimes \mathbb{R}^3$. Letting e_1, e_2, e_3 be the standard basis of $\mathbb{R}^3$, the space $\Lambda^2((\mathbb{R}^3)^*) \otimes \mathbb{R}^3$ has a basis $\{(e_i^* \wedge e_j^*) \otimes e_k\}$ where $1 \le i < j \le 3$ and $1 \le k \le 3$. Identify the cross product in terms of this basis.

Problem 3. Define

$$\begin{aligned} f \colon S^1 &\to \mathbb{R}^2 \\ (x, y) &\mapsto (x, y), \end{aligned}$$

and let $\omega = x^2\, dx \in \Omega^1(\mathbb{R}^2)$. Compute $\int_{S^1} f^*\omega$.

Problem 4. Let $S^n(r) = \{x \in \mathbb{R}^{n+1} \mid |x| = r\}$ be the n-sphere of radius r centered at the origin, and let $D^{n+1}(r) := \{x \in \mathbb{R}^{n+1} \mid |x| \le r\}$ be the $(n+1)$-dimensional closed ball of radius r centered at the origin. Define $\omega \in \Omega^{n-1}\mathbb{R}^n$ by

$$\omega = \frac{1}{r} \sum_{i=1}^{n+1} (-1)^{i-1} x_i\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1}.$$

It turns out that the n-dimensional volume of $S^n(r)$ is

$$\sigma_n(r) := \int_{S^n(r)} \omega.$$

Let $\tau_{n+1}(r)$ be the volume of $D^{n+1}(r)$. Prove that $\sigma_n(r) = \frac{n+1}{r}\tau_{n+1}(r)$. For instance, in the case $n = 1$, we have $\frac{2}{r} \cdot \pi r^2 = 2\pi r$, and in the case $n = 2$, we have $\frac{3}{r} \cdot \frac{4}{3}\pi r^3 = 4\pi r^2$.

Fun fact (nothing to turn in here): note that if we use Fubini's theorem, we can write $\tau_{n+1}(r) = \int_{t=0}^{r} \sigma_n(t)$, and it follows by the fundamental theorem of calculus that $\tau_n'(r) = \sigma_n(r)$.

Problem 5. (This is Exercise 11.4 in the latest version of our course notes.) Let M be a manifold. We defined a product on de Rham cohomology classes as follows:

$$\begin{aligned} \wedge \colon H^r M \times H^s M &\to H^{r+s} M \\ ([\omega], [\eta]) &\mapsto [\omega \wedge \eta]. \end{aligned}$$

We must show that this map is well-defined. In class, we showed that if ω and η are cocycles, then so is $\omega \wedge \eta$, (i.e., $d\omega = 0$ and $d\eta = 0$ implies $d(\omega \wedge \eta) = 0$). Hence, $\omega \wedge \eta$ represents a cohomology class. It remains to be shown that our product does not depend on the choice of representatives for $[\omega]$ and $[\eta]$. So, given $\mu \in \Omega^{r-1}M$ and $\nu \in \Omega^{s-1}M$, we must show that $[(\omega + d\mu) \wedge (\eta + d\nu)] = [\omega \wedge \eta]$. Find $\tau \in \Omega^{r+s-1}M$ such that

$$d\tau = ((\omega + d\mu) \wedge (\eta + d\nu)) - \omega \wedge \eta.$$