Problem 1. For each of the following functions, compute $f^*(d\omega)$ and $d(f^*\omega)$ longhand to verify they are equal.

(a) $f(u,v) = (u, uv^2, u^2 + uv)$ and $\omega = y\,dx + x^2\,dy + 3\,dz \in \Omega^1\mathbb{R}^3$.
(b) $f(u,v) = uv$ and $\omega = t\,dt \in \Omega^1\mathbb{R}$.

Problem 2. Suppose that ω is a 1-form on $\mathbb{P}^1$. Let (U_x, ϕ_x) and (U_y, ϕ_y) denote the two standard charts for $\mathbb{P}^1$. Say $f(t)\,dt$ is ω in (U_x, ϕ_x) coordinates and $g(s)\,ds$ is ω in (U_y, ϕ_y) coordinates. On the overlap, $U_x \cap U_y$, this gives two representations for ω, so it make sense to compare them.

(a) By pulling back, we have $f(t)\,dt = (\phi_y \circ \phi_x^{-1})^*(g(s)\,ds)$. What is the relation between f and g in terms of the coordinate t on U_x?
(b) In light of your answer to part (a), construct a nonzero (globally defined) 1-form on $\mathbb{P}^1$. There are many possible solutions. The key will be defining f at $t = 0$ in a way that is compatible with the relation you found in the first part of this problem—taking a limit as $t \to 0$ places a restriction on g. Your resulting form must be smooth.

Problem 3. For $i = 1, \ldots, n$, suppose we have curves $\alpha_i \colon [0,1] \to \mathbb{R}^n$. Let $\beta(t) := \{\alpha_1(t), \ldots, \alpha_n(t)\}$, and suppose that $\beta(t)$ is a basis for $\mathbb{R}^n$ for all $t \in [0,1]$. Prove, explaining your reasoning, that $\beta(0)$ and $\beta(1)$ have the same orientation.

Problem 4. Consider the atlas $\mathfrak{U} := \{(U_x, h), (U_y, k)\}$ for $\mathbb{P}^1$ given by

$$\begin{aligned} h\colon U_x := \{(x,y) \in \mathbb{P}^1 : x \neq 0\} &\to \mathbb{R} \\ (x,y) &\mapsto y/x \end{aligned}$$

$$\begin{aligned} k\colon U_y := \{(x,y) \in \mathbb{P}^1 : y \neq 0\} &\to \mathbb{R} \\ (x,y) &\mapsto -x/y. \end{aligned}$$

(a) Show that $\mathfrak{U}$ is an orienting atlas.
(b) Let $A_1 = h^{-1}([-1,1])$ and $A_2 = k^{-1}([-1,1])$. Show that $A_1 \cup A_2 = \mathbb{P}^1$ with intersection of measure 0.
(c) Define $\omega \in \Omega^1\mathbb{P}^1$ whose local description with respect to h is

$$\omega|_{U_x}(t) = \frac{1}{t^2+1}\,dt.$$

  (i) By pulling back along $h \circ k^{-1}$, find the local form for ω on U_y.
  (ii) Compute $\int_{\mathbb{P}^1} \omega$ using A_1 and A_2 and following the definition in the book.