Problem 1. Think of $\mathbb{P}^1$ as the set of lines through the origin in $\mathbb{R}^2$, and define

$$E = \left\{ (\ell, p) \in \mathbb{P}^1 \times \mathbb{R}^2 \mid p \in \ell \right\}.$$

In other words, a point in $E$ consists of a pair $(\ell, x)$ where $\ell$ is a point in $\mathbb{P}^1$, and $p \in \mathbb{R}^2$ is a point in the line represented by $\ell$. There is a projection mapping $\pi\colon E \to \mathbb{P}^1$ defined by $\pi(\ell, p) = \ell$. We would like to show that $E$ has the structure of a line bundle over $\mathbb{P}^1$. Let $(U_i, \phi_i)$ for $i = 1, 2$ be the standard charts for $\mathbb{P}^1$. For $i = 1, 2$, we would like to define homeomorphisms $\Psi_i\colon \pi^{-1}(U_i) \to U_i \times \mathbb{R}$ so that the following holds:

(i) If $\Psi_i$ is restricted to any fiber $E_\ell := \pi^{-1}(\ell)$ for $\ell \in U_i$, then it induces a linear isomorphism $E_\ell \to \mathbb{R}^1$.

(ii) A manifold structure on $E$ is given by, for $i = 1, 2$,

$$\psi_i\colon \pi^{-1}(U_i) \xrightarrow{\Psi_i} U_i \times \mathbb{R} \xrightarrow{\phi_i \times \mathrm{id}} \mathbb{R} \times \mathbb{R}.$$

First consider the case $i = 1$, and consider a point $(\ell, p) \in \pi^{-1}(U_1)$. There is a unique representative for the homogeneous coordinates for $\ell$ of the form $(1, t)$ for some $t \in \mathbb{R}$. (If $\ell = (x, y)$, then $t = y/x$.) Then, since $p \in \ell$, we can write $p = \lambda(1, t)$ for a uniquely determined $\lambda \in \mathbb{R}$. Define

$$\Psi_1(\ell, p) := (\ell, \lambda) \in U_1 \times \mathbb{R}^1.$$

We have

$$E_\ell = \{((1, t), \lambda(1, t)) \mid \lambda \in \mathbb{R}\},$$

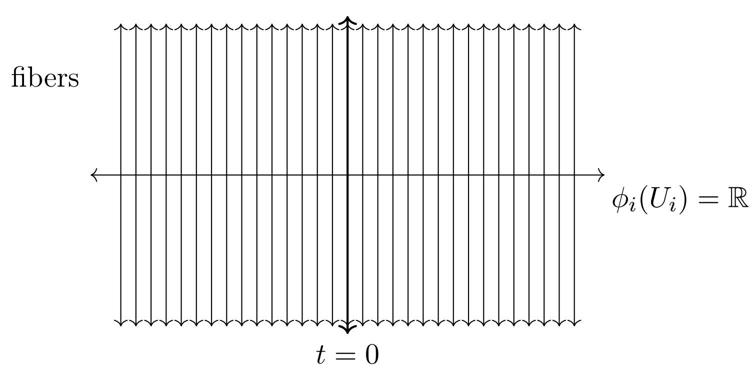
which has linear structure defined by

$$\alpha((1, t), \lambda(1, t)) + ((1, t), \lambda'(1, t)) := ((1, t), (\alpha\lambda + \lambda')(1, t)).$$

So it is clear that $\Psi_1$ restricted to any fiber is a linear isomorphism.

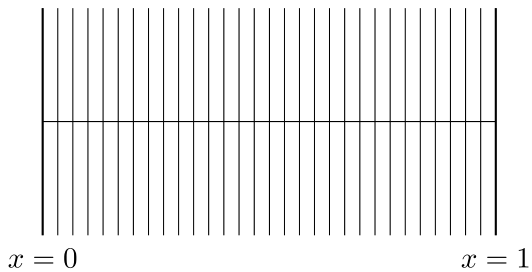
**Problem:** For $(t, \lambda) \in \psi_1(\pi^{-1}(U_1) \cap \pi^{-1}(U_2)) = (\mathbb{R} \setminus \{0\}) \times \mathbb{R}$, compute $(\psi_2 \circ \psi_1^{-1})(t, \lambda)$. This will give a formula for the transition function from which it should be clear the charts are differentiably related.

Problem 2. The vector bundle just constructed comes from gluing together two copies of $\mathbb{R} \times \mathbb{R}^1$ (since $\phi_i(U_i) = \mathbb{R}$):



Try to picture the gluing and describe the resulting shape. (Warning: when considering a point $(t, \lambda)$, pay attention to the sign of $t$.)

PROBLEM 3. Define the Möbius bundle $M$ as the quotient of $[0, 1] \times \mathbb{R}$ by the equivalence $(0, t) \sim (1, -t)$. The points of the form $(x, 0)$ in $M$ form the circle $S^1$, and there is a well-defined projection mapping $M \to S^1$. Geometrically, the Möbius bundle is a corkscrew of lines attached to $S^1$, and it would not be too hard to rigorously show that $M$ is a line bundle over $S^1$. Prove that $M \to S^1$ is not trivial by showing it has no global non-vanishing section (one that is never 0). The following picture may be helpful:




PROBLEM 4. In the following, we refer to the standard charts $(U_i, \phi_i)$ on $\mathbb{P}^n$.

(a) Compute the transition function from $U_1$ to $U_2$ for $T\mathbb{P}^1$ (really $\pi^{-1}(U_1)$ to $\pi^{-1}(U_1)$).
(b) Compute the transition function from $U_1$ to $U_2$ for $T\mathbb{P}^2$.

PROBLEM 5. Define

$$\begin{aligned} f\colon \mathbb{R}^2 &\to \mathbb{R}^4 \\ (x, y) &\mapsto (x^2, 2x + y, y^4, xy) \end{aligned}$$

(a) Let $\omega = y_1\, dy_1 \wedge dy_2 + y_1 y_3\, dy_3 \wedge dy_4 \in \Omega^2\mathbb{R}^4$. Compute $f^*\omega$ and express your answer in terms of the standard basis $\{dx \wedge dy\}$. for $\Omega^2\mathbb{R}^2$.
(b) Consider the vector field $v = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ on $T\mathbb{R}^2$. Let $p = (1, 1)$, and compute $df_{(1,1)}(v)$ in terms of the standard basis for $T_{f(p)}\mathbb{R}^4$. (We call $df(v)$ the *push forward* of the vector field $v$ and sometimes denote it by $f_*v$. We push forward vector fields and pull back forms.)

PROBLEM 6. Consider the polar coordinates mapping

$$\begin{aligned} f\colon I := (0, 1) \times (0, 2\pi) &\to \mathbb{R}^2 \\ (r, \theta) &\mapsto (r\, \cos\theta, r\, \sin\theta) \end{aligned}$$

and the "volume form", $\omega := dx \wedge dy \in \Omega^2\mathbb{R}^2$. Compute $f^*\omega \in \Omega^2 I$ using the coordinates $(r, \theta)$ on $I$.