PROBLEM 1. Let V be a vector space over a field $\Bbbk$, and let $v_1, \ldots, v_n$ be a basis for V. For each $i = 1, \ldots, n$, define $v_i^*\colon V \to \Bbbk$ by

$$\begin{aligned} v_i^*\colon V &\to \mathbb{R} \\ v_j &\mapsto \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Prove that $v_1^*, \ldots, v_n^*$ is a basis for $V^* := \hom(V, \Bbbk)$.

PROBLEM 2. Let $L\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear function. The function L is represented by the matrix A whose j-th column is $L(e_j)$, the image of the j-th standard basis vector. Let $e_1, \ldots, e_n$ and $f_1, \ldots, f_m$ be the standard bases for $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Choosing bases dual to the standard bases, show that the matrix representing $L^*\colon (\mathbb{R}^m)^* \to (\mathbb{R}^n)^*$ is A^t, the transpose of A.

PROBLEM 3. Recall that the *determinant*, det, of a square matrix over a field $\Bbbk$ is the unique multilinear alternating function of its rows that sends the identity matrix to 1. Let $e_1, \ldots, e_n$ be the standard basis of $\Bbbk^n$ and let $v_1, \ldots, v_n \in \Bbbk^n$.

(a) Prove that $v_1 \wedge \cdots \wedge v_n = \det(v_1, \ldots, v_n)\, e_1 \wedge \cdots \wedge e_n$.
(b) For every $\ell \geq 1$, show that $v_1, \ldots, v_\ell \in \Bbbk^n$ are linearly dependent if and only if $v_1 \wedge \cdots \wedge v_\ell = 0$.

PROBLEM 4. Let $V = W = \mathbb{R}^3$, and let e_1, e_2, e_3 be the standard basis for V and f_1, f_2, f_3 be the standard basis for W. Of course, $e_i = f_i$ for all i, but it is useful to have separate notation.

(a) From the section on pullbacks in the notes, we have the isomorphism

$$\begin{aligned} \Lambda^\ell W^* &\to (\Lambda^\ell W)^* \\ \phi_1 \wedge \cdots \wedge \phi_\ell &\mapsto [w_1 \wedge \cdots \wedge w_\ell \mapsto \det(\phi_i(w_j))] \end{aligned}$$

for each $\ell \in \mathbb{N}$. An element of $(\Lambda^\ell W)^*$ is a linear mapping $\Lambda^\ell W \to \mathbb{R}$, and by the universal property of exterior products, this mapping corresponds to a multilinear alternating mapping $W \times \cdots \times W \to \mathbb{R}$, i.e., to an element of $\mathrm{Alt}^\ell(W)$—an alternating ℓ-form. Let

$$\omega = f_1^* \wedge f_3^* \in \Lambda^2 W^*,$$

and let $\tilde{\omega} \in \mathrm{Alt}^2(W)$ be the corresponding bilinear alternating 2-form. Describe $\tilde{\omega}$ by calculating the images of (f_i, f_j) for $1 \leq i < j \leq 3$.

(b) Let $L\colon V \to W$ be a linear mapping with matrix $A = (a_{ij})$ relative to the standard bases (using notation as in part (a)). From the notes, we have the following formulations for pullback mappings:

$$\begin{array}{cccccc} L^*\colon \Lambda^\ell W^* & \to & \Lambda^\ell V^* & \to & (\Lambda^\ell V)^* \\ \phi_1 \wedge \cdots \wedge \phi_\ell & \mapsto & L^*\phi_1 \wedge \cdots \wedge L^*\phi_\ell & \mapsto & [v_1 \wedge \cdots \wedge v_\ell \mapsto \det((\phi_i \circ L)(v_j))] \end{array}$$

for all $\ell \in \mathbb{N}$. We would like to write

$$L^*\omega = \alpha\, e_1^* \wedge e_2^* + \beta\, e_1^* \wedge e_3^* + \gamma\, e_2^* \wedge e_3^*$$

where α, β, γ are functions of the entries of $A = (a_{ij})$. Solve this in two ways:

(i) First use the mapping

$$\begin{array}{rcl} L^*\colon \Lambda^\ell W^* & \to & \Lambda^\ell V^* \\ \phi_1 \wedge \cdots \wedge \phi_\ell & \mapsto & L^*\phi_1 \wedge \cdots \wedge L^*\phi_\ell \end{array}$$

[You will need to compute $L^* f_1^*$ and $L^* f_3^*$ in terms of the e_i^* and the entries of A.]

(ii) Next use the composite mapping

$$\begin{array}{rcl} L^*\colon \Lambda^\ell W^* & \to & (\Lambda^\ell V)^* \\ \phi_1 \wedge \cdots \wedge \phi_\ell & \mapsto & [v_1 \wedge \cdots \wedge v_\ell \mapsto \det((\phi_i \circ L)(v_j))] \end{array}$$

[To find the coefficients, compute $(L^*\omega)(e_i \wedge e_j)$ for each $i < j$.]

Problem 5. Describe a natural mapping $f\colon V \to (V^*)^*$, i.e., one that does not depend on a choice of basis, and prove it is injective. (FYI: Recall that if W is finite-dimensional, then $W \approx W^*$ via a choice of basis, and in particular, $\dim W = \dim W^*$. Therefore, if V is finite-dimensional, since f is injective, it is an isomorphism.)