PROBLEM 1. Let M be a manifold, and let $\mathcal{E}_p(M)$ be the ring of germs of functions at $p \in M$. Let $v\colon \mathcal{E}_p(M) \to \mathbb{R}$ be a derivation, i.e., a linear mapping satisfying the product rule. Let f be a constant real-valued function defined on some neighborhood about p. What is $v(f)$? Give an explanation that follows directly from the properties of a derivation—do not take coordinates.

PROBLEM 2. Consider the projective plane $\mathbb{P}^2$ with homogeneous coordinates (x, y, z), and let $p = (1, 1, 1) \in \mathbb{P}^2$. Define

$$f(x, y, z) = \frac{x}{y}.$$

(a) Show that f is a well-defined function in a neighborhood of the point p.

(b) Consider the curve $\alpha(t) = (1 + t, 1 + t^2, 1 + t^3) \in \mathbb{P}^2$ for t in a small open interval about 0. The curve α determines a derivation, v_α, of germs at p. What is $v_\alpha(f)$?

(c) Consider the standard chart (U_x, ϕ_x) at p, i.e., $U_x = \{(x, y, z) \in \mathbb{P}^2 \mid x \neq 0\}$ with $\phi_x(x, y, z) = (y/x, z/x)$. Let (u, v) denote the coordinates on $\mathbb{R}^2$ here. The standard basis for $T_p\mathbb{P}^2$ with respect to this chart is

$$\left(\frac{\partial}{\partial u}\right)_p, \left(\frac{\partial}{\partial v}\right)_p.$$

What are the coordinates of the tangent vector determined by α in terms of this basis? [Hint: think in terms of $T_p^{\text{phy}}\mathbb{P}^2$. What is the tangent vector determined by α? What is the standard basis?]

(d) Repeat the previous exercise, (c), with respect to the chart (U_y, ϕ_y).

(e) Show that your solution to (c) is sent to your solution to (d) by the derivative of the change of coordinates mapping $\phi_y \circ \phi_x^{-1}$.

PROBLEM 3. Show that the composition

$$T_p^{\text{phys}}M \xrightarrow{\Phi_3} T_p^{\text{geom}}M \xrightarrow{\Phi_1} T_p^{\text{alg}}M \xrightarrow{\Phi_2} T_p^{\text{phys}}M$$

is the identity. (Start with a physically-defined tangent vector v. As part of your explanation, describe $\Phi_3(v)$ and $(\Phi_1 \circ \Phi_3)(v)$.)

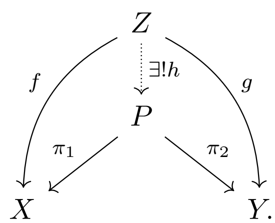
PROBLEM 4. Consider the mapping

$$\begin{aligned} f\colon \mathbb{R}^2_{>0} &\to \mathbb{P}^3 \\ (x, y) &\mapsto (1, x + xy, 3 + 5y, x^2 + 2y). \end{aligned}$$

Here the domain is the positive quadrant of $\mathbb{R}^2$. Let $p = (1, 2)$. What is the induced mapping on tangent spaces $df_p\colon T_p\mathbb{R}^2_{>0} \to T_{f(p)}\mathbb{P}^3$? Choose the standard chart $(\mathbb{R}^2_{>0}, \text{id})$ on the domain and the standard chart (U_1, ϕ_1) on the codomain, where U_1 is the set of points in $\mathbb{P}^3$ whose first homogeneous coordinate is nonzero and ϕ_1 is the usual corresponding chart map for $\mathbb{P}^3$. Your answer will then take the form of a linear function $\mathbb{R}^2 \to \mathbb{R}^3$.

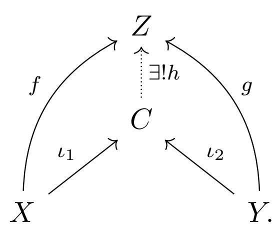
Problem 5. (Product and coproduct in the category of sets.) Let X and Y be sets.

(a) Describe a set P and mappings $P \xrightarrow{\pi_1} X$ and $P \xrightarrow{\pi_2} Y$ satisfying the following universal property: Given a set Z and (set) functions $f\colon Z \to X$ and $g\colon Z \to Y$, there exists a unique function $h\colon Z \to P$ making the following diagram commute:



Describe the mapping h.

(b) The previous part of this problem defines the *product* in the category of sets. Repeat this problem but for the *coproduct* in the category of sets. Describe a set C and mappings $X \xrightarrow{\iota_1} C$ and $Y \xrightarrow{\iota_2} C$ satisfying the following universal property: Given a set Z and functions (of sets) $f\colon X \to Z$ and $g\colon Y \to Z$, there exists a unique function $h\colon C \to Z$ making the following diagram commute:



Describe the mapping h.

Problem 6. Let $V = \mathbb{R}^2$ and let $W = \mathbb{R}^3$ with standard bases e_1, e_2 and f_1, f_2, f_3, respectively.

(a) Express $(2,4) \otimes (5,2,3)$ as a linear combination of the $e_i \otimes f_j$.

(b) By letting $v = (\alpha_1, \alpha_2) = \alpha_1 e_1 + \alpha_2 e_2$ and $w = (\beta_1, \beta_2, \beta_3) = \beta_1 f_1 + \beta_2 f_2 + \beta_3 f_3$, directly show, using linear algebra, that there are no $v \in V$ and $w \in W$ such that $v \otimes w = e_1 \otimes f_1 + e_2 \otimes f_2$.