PROBLEM 1. Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ (where $\mathbb{R}^n$ and $\mathbb{R}^m$ are given their usual topology having a bases of open balls). Show that f is continuous (as a mapping of topological spaces, i.e., the inverse image of each open set is open) if and only if it is continuous in the usual ϵ-δ sense. (Recall that the standard topology on $\mathbb{R}^n$ has the set of open balls as basis, i.e., $U \subseteq \mathbb{R}^n$ is open if and only if for all $p \in U$ there exists an open ball B such that $p \in B \subseteq U$.)

PROBLEM 2. Show that the projection mapping $\pi\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ sending a point to its equivalence class (modulo scaling) is differentiable.

PROBLEM 3.

(a) Show that the following two charts on $\mathbb{R}$ are not differentiably compatible: $(\mathbb{R}, \mathrm{id})$ where $\mathrm{id}(x) = x$ and $(\mathbb{R}, h)$ where $h(x) = x^3$.
(b) Let $\mathcal{D}_1$ be the differentiable structure on $\mathbb{R}$ containing $(\mathbb{R}, \mathrm{id})$, and let $\mathcal{D}_2$ be the differentiable structure on $\mathbb{R}$ containing $(\mathbb{R}, h)$. Consider the manifolds $M_1 = (\mathbb{R}, \mathcal{D}_1)$ and $M_2 = (\mathbb{R}, \mathcal{D}_2)$. Show that even though their differentiable structures are not compatible that, nevertheless, the mapping $f\colon M_2 \to M_1$ given by $f(x) = x^3$ is a diffeomorphism. (You need to show f and its inverse are differentiable as mapping of manifolds.)

PROBLEM 4. Let $\mathcal{K}_p(M)$ denote the set of differentiable curves in M that pass through p at 0:

$$\mathcal{K}_p(M) = \{\alpha\colon (-\varepsilon, \varepsilon) \to M \mid \alpha \text{ is differentiable, } \varepsilon > 0, \text{ and } \alpha(0) = p\}.$$

Two such curves $\alpha, \beta \in \mathcal{K}_p(M)$ are called *tangentially equivalent*, denoted $\alpha \sim \beta$, if

$$(h \circ \alpha)'(0) = (h \circ \beta)'(0) \in \mathbb{R}^n$$

for some chart (U, h) around p. We call the equivalence classes $\mathcal{K}_p(M)$ the *(geometrically-defined) tangent vectors* of M at p and call the quotient

$$T_p^{\mathrm{geom}} M := \mathcal{K}_p(M)/\sim$$

the *(geometrically-defined) tangent space* to M at p. Show that $\sim$ is independent of the choice of chart (U, h). (Use Jacobian matrices to make your proof precise.)

PROBLEM 5. To define a linear structure on $T_p^{\mathrm{geom}} M$ we fixed a chart (U, h) at p and defined

$$\begin{aligned} \phi\colon T_p^{\mathrm{geom}} M &\to \mathbb{R}^n \\ [\alpha] &\mapsto (h \circ \alpha)'(0). \end{aligned}$$

The fact that ϕ is well-defined and injective follows immediately from the definition of $\sim$. Prove that ϕ is surjective (and, hence, a bijection). [This exercise is important for us since once we know ϕ is a bijection, we can just *define* the linear structure on $T_p^{\mathrm{geom}} M$ to be the unique linear structure that makes ϕ a linear isomorphism.]

Problem 6. Let $f\colon \mathbb{R}^n \to \mathbb{R}$, and let $p \in \mathbb{R}^n$. Prove that

$$\frac{d}{dt} f(tx + (1-t)p) = \sum_{i=1}^{n} \left[\frac{\partial f}{\partial x_i}(tx + (1-t)p)(x_i - p_i) \right].$$

Your proof should explain exactly how the chain rule (as stated in our text) applies by explicitly stating the functions we are composing and showing how the above formulas relate to Jacobian matrices.