Math 341 for Friday, Week 2

Problem 1. The *standard atlas* $\{(U_1, \phi_1), (U_2, \phi_2)\}$ for $\mathbb{P}^1$ has two "pages", $\phi_1(U_1)$ and $\phi_2(U_2)$, each equal to $\mathbb{R}$. (See our notes or the lecture for Wednesday, Week 2 for definitions.) Compute the transition function from U_1 to U_2 to see how these pages are glued together to form $\mathbb{P}^1$.

Problem 2. (For us, *atlas* always means *differentiable atlas*, i.e., that the transition functions are differentiable.) Let V be an n-dimensional vector space over $\mathbb{R}$. A choice of ordered basis $\mathbb{B}$ for V gives a linear isomorphism

$$h_{\mathbb{B}} \colon V \to \mathbb{R}^n.$$

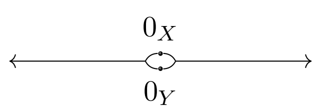
Give V a topology by declaring a set open in V iff its image under $h_{\mathbb{B}}$ is open in $\mathbb{R}^n$. Then $(V, h_{\mathbb{B}})$ is a chart at each point in V, hence, $\mathfrak{A} := \{(V, h_{\mathbb{B}})\}$ is an atlas containing exactly one chart. (It's differentiable since the only transition function is the identity mapping from $\mathbb{R}^n$ to itself.) Let $\mathbb{B}'$ be any other ordered basis, and consider the corresponding atlas $\mathfrak{A}' = \{(V, h'_{\mathbb{B}})\}$. Show that $\mathfrak{A} \cup \mathfrak{A}'$ is a *differentiable* atlas.

Point of this problem: The maximal atlas containing $\mathfrak{A}$ determines a differentiable structure on V and makes V a manifold. The choice of a different ordered basis determines the same differentiable structure. In this sense, V has a canonical manifold structure. (Make sure to review the definition of *differentiable structure* in our text.)

Problem 3. Let X and Y be disjoint copies of $\mathbb{R}$ with the usual topology. Put a topology on the disjoint union $X \cup Y$ by defining a subset $S \subseteq X \cup Y$ to be open if $S \cap X$ and $S \cap Y$ are both open. For $x \in X$ and $y \in Y$, say $x \sim y$ if $x = y \neq 0$ as real numbers. Define $L = (X \cup Y)/\!\sim$ as a topological space with the quotient topology. In other words, if

$$\begin{aligned} \pi \colon X \cup Y &\to L \\ a &\mapsto [a], \end{aligned}$$

where $[a] = \{b \in X \cup Y : a \sim b\}$, then a subset $U \subseteq L$ is open if and only if $\pi^{-1}(U)$ is open in $X \cup Y$ (which means $\pi^{-1}(U) \cap X$ and $\pi^{-1}(U) \cap Y$ are open a subsets of the real numbers). Thus, L is essentially the real number line with two origins, 0_X from X and 0_Y from Y. Provide proofs or counterexamples.



(a) Is L locally Euclidean?

(b) Is L Hausdorff?