Math 341 Homework for Friday, Week 1

Problem 1. Let $X = Y = Z = W = \mathbb{R}^2$ with coordinates (x_1, x_2), (y_1, y_2), (z_1, z_2), and (w_1, w_2) respectively. Gluing instructions are given by making the following identifications:

$$\begin{aligned} y_1 &= 1/x_1 \quad \text{and} \quad y_2 = x_2, \\ z_1 &= x_1 \quad \text{and} \quad z_2 = 1/x_2, \\ w_1 &= 1/x_1 \quad \text{and} \quad w_2 = 1/x_2. \end{aligned}$$

These identifications imply others. For instance, to see how Z is glued to W, we have

$$z_1 = x_1,\ z_2 = 1/x_2,\ w_1 = 1/x_1,\ w_2 = 1/x_2 \quad \Rightarrow \quad z_1 = 1/w_1,\ z_2 = w_2.$$

(a) What shape A result from using these gluing instructions to glue X to Y. (No explanation required.)

(b) What shape B results from gluing Z to W? (No explanation required.)

(c) Gluing all four copies of $\mathbb{R}^2$ together now amounts to gluing A to B. Draw a picture showing how A is glued to B. What is the resulting shape?

Problem 2. Let

$$\begin{aligned} f\colon \mathbb{R}^3 &\to \mathbb{R} \\ (x, y, z) &\mapsto x^2 + y^2 + z^2. \end{aligned}$$

(a) Describe the level sets of f.

(b) Compute the gradient of f and use it to compute a vector perpendicular to the level set $f = 14$ at the point $p = (1, 2, 3)$.

(c) What is the maximal rate of change of f moving away from the point $p = (1, 2, 3)$?

Problem 3. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a differentiable function, and let $p \in \mathbb{R}^n$. Let $C\colon \mathbb{R} \to \mathbb{R}^n$ be a differentiable parametrized curve whose image sits in a level set of f and passing through p at time $t = 0$. Use the chain rule to prove that $\nabla f(p) \cdot C'(0) = 0$. (This proves that the gradient at p is perpendicular to the level set containing p.)

Problem 4. Let

$$\begin{aligned} C\colon \mathbb{R} &\to \mathbb{R}^3 \\ t &\mapsto (t, t^2, t^3). \end{aligned}$$

(a) Letting the coordinates on $\mathbb{R}^3$ be x, y, z, draw pictures of the projection of the image of C to the three coordinate planes, i.e., the x, y-plane, the x, z-plane and the y, z-plane.

(b) To the best of your ability, sketch the image of C in $\mathbb{R}^3$.

(c) Compute a parametrization $\psi(t)$ of the tangent line to the curve C at time $t = 1$ so that $\psi(1) = C(1)$.

Problem 5. Let

$$S\colon \mathbb{R}^2 \to \mathbb{R}^3$$
$$(x, y) \mapsto (x, y, x^2 + y^2).$$

(a) Sketch the image of the parametrize surface S.

(b) Compute $JS(1, 2)$, the Jacobian of S at the point $p = (1, 2)$.

(c) The columns of $JS(1, 2)$ can be thought of as tangent vectors to the surface S at $p = (1, 2)$ (but translated to the origin). Give a parametrization $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ of the span of these vectors.

(d) Give a parametrization $A\colon \mathbb{R}^2 \to \mathbb{R}^3$ of the tangent plane to S at $p = (1, 2)$ so that $A(0, 0) = S(1, 2)$.

PROBLEM 6. Going back to Problem 4, consider the *twisted cubic curve*, $C(t) = (t, t^2, t^3)$. Give a parametrization $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ of the surface formed by the union of all of the tangent lines to C. (Can you visualize the image? The twisted cubic curve sits in the image, and the surface has interesting behavior along it.)