Differential Equations Recipes

I. Separable.

A. $\boxed{p(y)\dfrac{dy}{dt} = q(t)}$

Solution: $\int p(y)\,dy = \int q(t)\,dt + c$.

B. $\boxed{\dfrac{dy}{dt} = F\left(\dfrac{y}{t}\right)}$

Solution: Let $v = \frac{y}{t}$. Then $\frac{dy}{dt} = \frac{d(vt)}{dt} = t\frac{dv}{dt} + v$. Hence, $\frac{dy}{dt} = F\left(\frac{y}{t}\right) \Rightarrow t\frac{dv}{dt} + v = F(v) \Rightarrow \int \frac{dv}{F(v)-v} = \int \frac{dt}{t}$.

II. Exact.

A. $\boxed{M(t,y) + N(t,y)\frac{dy}{dt} = 0,\ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}}$

Solution: Suppose we have a function of two variables, Φ, such that $\Phi(t,y) = 0$. Then by the chain rule:

$$\frac{d\Phi}{dt} = \frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial y}\frac{dy}{dt} = 0.$$

If it turned out that $\frac{\partial \Phi}{\partial t} = M(t,y)$ and $\frac{\partial \Phi}{\partial y} = N(t,y)$, we would be in luck because $\Phi(t,y) = 0$ would give the solution y (at least implicitly). To find such a Φ we should integrate M with respect to t, then take its partial with respect to y and set it equal to N.

Here is an example: Solve $3t^2 - 2y^2 + (1-4ty)\frac{dy}{dt} = 0$. This equation is exact with $M(t,y) = 3t^2 - 2y^2$ and $N(t,y) = 1 - 4ty$. Integrate M with respect to t:

$$\int M(t,y)\,dt = \int 3t^2 - 2y^2\,dt = t^3 - 2ty^2 + f(y).$$

This is our candidate for Φ. We need to find the appropriate function f. Take the partial with respect to y to get $-4ty + f'(y)$. Setting this equal to N, we find that $f'(y) = 1$; so $f(y) = y + c$. The solution is $t^3 - 2ty^2 + y = -c$. Of course, this is cheating a little since y is only defined implicitly as a function of t.

B. $\boxed{M(t,y) + N(t,y)\dfrac{dy}{dt} = 0}$

Solution: Sometimes an equation can be made exact by multiplying through by an integrating factor $\rho(t,y)$: $\rho(t,y)M(t,y) + \rho(t,y)N(t,y)\frac{dy}{dt} = 0$.

Example: Solve $ty^2 + 4t^2y + (3t^2y + 4t^3)\frac{dy}{dt} = 0$. This equation is not exact. Look for an integrating factor of the form $\rho(t,y) = t^m y^n$. We want

$$\frac{\partial}{\partial y}(t^{m+1}y^{n+2} + 4t^{m+2}y^{n+1}) = \frac{\partial}{\partial t}(3t^{m+2}y^{n+1} + 4t^{m+3}y^n).$$

Equating coefficients shows that $m = -1$, $n = 1$. So $\rho(t,y) = \frac{y}{t}$ is the integrating factor. The problem now is to solve $y^3 + 4ty^2 + (3ty^2 + 4t^2y)\frac{dy}{dt} = 0$, which is exact.

III. Linear, first order.

A. $\boxed{\dfrac{dy}{dt} + p(t)y = q(t) \quad \left(\text{or } a(t)\frac{dy}{dt} + b(t)y = c(t)\right)}$

Solution: Use the integrating factor $e^{\int p(t)\,dt}$:

$$e^{\int p(t)\,dt}\left(\frac{dy}{dt} + p(t)y\right) = e^{\int p(t)\,dt}q(t).$$

The left-hand side of the above equation is $\dfrac{d(e^{\int p(t)\,dt}y)}{dt}$. Hence,

$$e^{\int p(t)\,dt}y = \int q(t)e^{\int p(t)\,dt}\,dt + c.$$

B. $\boxed{\text{(Bernoulli)} \quad \dfrac{dy}{dt} + p(t)y = q(t)y^n}$

Solution: Substitute $u = y^{1-n}$ and solve using the preceding method.

IV. Linear, homogeneous, constant coefficients.

$\boxed{P(D)y = 0 \quad D = \text{ differential operator, } P = \text{ polynomial}}$

Solution: The main idea is to look for solutions of the form $y = e^{rt}$. One finds $0 = P(D)e^{rt} = P(r)e^{rt}$ iff $P(r) = 0$. In this context, $P(r)$ is called the *characteristic polynomial* of the equation. Examples follow.

1. $y'' - y' - 2y = 0 = (D^2 - D - 2)y$

$$r^2 - r - 2 = (r-2)(r+1) \Rightarrow r = 2 \text{ or } r = -1.$$

   Solution: $y = Ae^{2t} + Be^{-t}$

2. $y'' - 2y' + 5y = 0 = (D^2 - 2D + 5)y$

$$r^2 - 2r + 5 = 0 \Rightarrow r = 1 + 2i \text{ or } r = 1 - 2i.$$

   Solution: $y = Ae^{(1+2i)t} + Be^{(1-2i)t}$.

   Recalling that $e^{(a+bi)t} = e^{at}\cos bt + e^{at}\,i\sin bt$ and adjusting the constants we get the real solution: $y = Ae^t\cos 2t + Be^t\sin 2t$

3. $y'' + 4y' + 4y = 0 = (D^2 + 4D + 4)y$

$$r^2 + 4r + 4 = 0 = (r+2)^2 \Rightarrow r = -2 \text{ (multiplicity 2)}.$$

   Solution: $y = Ae^{-2t} + Bte^{-2t}$

4. $D^2(D+2)^3(D-3)y = 0$

   Solution: $y = A + Bt + Ce^{-2t} + Dte^{-2t} + Et^2e^{-2t} + Fe^{3t}$.

V. Method of undetermined coefficients. (Linear, constant coefficients, *non-homogeneous*)

$$\boxed{P(D)y = f(t)}$$

Solution: If we can find a particular solution, y_p, the general solution is then the sum of the general solution to the associated homogeneous equation, $P(D)y = 0$, and y_p. (Clearly, if y is a solution to the associated homogeneous equation, then $P(D)(y+y_p) = P(D)y + P(D)y_p = P(D)y_p$. Thus, we get another solution. There is a theorem that says all solutions have this form.)

Depending on the form of f, the solution will have the form given in the following list:

- If $f(t) = e^{rt}$, guess $y = ae^{rt}$.
- If $f(t)$ is a polynomial, guess $y = p(t) = \sum_{i=0}^{n} a_i t^i$ where p is a general polynomial of the same degree as f.
- If $f(t) = \cos \omega t$ or $f(t) = \sin \omega t$, guess $y = a\cos\omega t + b\sin\omega t$.
- if $f(t) = q(t)e^{rt}$ where q is a polynomial, guess $y = p(t)e^{rt}$ where p is a general polynomial of the same degree as q.
- If $f(t) = q(t)e^{rt}\cos\omega t$ or $f(t) = q(t)e^{rt}\sin\omega t$, guess $y = e^{rt}(u(t)\cos\omega t + v(t)\sin\omega t)$ where $u(t)$ and $v(t)$ are general polynomials of the same degree as q.

If the form you use happens to be a solution of the corresponding homogeneous equation, try multiplying it by t.

Examples:

1. $y'' - y' - 2y = 20e^{4t}$

   We look for a solution of the form ae^{4t}:

$$(ae^{4t})'' - (ae^{4t})' - 2ae^{4t} = 10ae^{4t}.$$

   Letting $a = 2$, we find the particular solution $y = 2e^{4t}$. The corresponding homogeneous equation, $y'' - y' - 2y = 0$ had characteristic polynomial $r^2 - r - 2 = (r-2)(r+1)$ with roots 2 and -1. So the general solution to the non-homogeneous equation is

$$y = 2e^{4t} + Be^{2t} + Ce^{-t}.$$

2. $y'' - 2y' + y = 6\sin 2t$

We guess a solution of the form $y = a\cos 2t + b\sin 2t$:

$$\begin{aligned}&(a\cos 2t + b\sin 2t)'' - 2(a\cos 2t + b\sin 2t)' + (a\cos 2t + b\sin 2t)\\ &\quad = (-3a - 4b)\cos 2t + (4a - 3b)\sin 2t.\end{aligned}$$

We want this to equal $6\sin 2t$, so we solve the system of equations

$$\begin{aligned}-3a - 4b &= 0\\ 4a - 3b &= 6\end{aligned}$$

and find $a = 24/25$ and $b = -18/25$. The corresponding homogeneous equation has characteristic polynomial $r^2 - 2r + 1 = (r-1)^2$ having one root, 1, with multiplicity 2. So the general solution to the original non-homogeneous equation is

$$y = \frac{24}{25}\cos 2t - \frac{18}{25}\sin 2t + Ce^t + Dte^t.$$

3. $y'' + 5y' + 6y = 4 - t^2$

   We guess a solution of the form $a + bt + ct^2$:

$$(a + bt + ct^2)'' + 5(a + bt + ct^2)' + 6(a + bt + ct^2) = (2c + 5b + 6a) + (10c + 6b)t + 6ct^2,$$

   forcing $c = -1/6$, $b = 5/18$, and $a = 53/108$. Combining this particular solution with the general solution to the corresponding homogeneous system gives the most general solution to the original equation:

$$y = \frac{58}{108} + \frac{5}{18}t - \frac{1}{6}t^2 + Ae^{-2t} + Be^{-3t}.$$

4. $y'' - y' - 2y = 5e^{-t}$

   We guess a solution of the form ae^{-t}. However, plugging this into the equation gives

$$(ae^{-t})'' - (ae^{-t}) - 2ae^{-t} = 0$$

   and there is no way to adjust the constant a to get $5e^{-t}$. The problem is that our guess is a solution to the associated homogeneous system. So our next guess is ate^{-t}:

$$(ate^{-t})'' - (ate^{-t}) - 2ate^{-t} = -3ae^{-t},$$

   and let $a = -5/3$. Adding the general solution to the associated homogeneous system gives the general solution to the original problem:

$$y = -\frac{5}{3}te^{-t} + Ae^{-t} + Be^{2t}.$$

VI. Second order.

A. $\boxed{H(t, y', y'') = 0}$

Solution: Substitute $\dfrac{dy}{dt} = v$, $\dfrac{d^2y}{dt^2} = \dfrac{dv}{dt}$.

B. $\boxed{H(y, y', y'') = 0}$

Solution: Substitute $\dfrac{dy}{dt} = v$, $\dfrac{d^2y}{dt^2} = \dfrac{dv}{dt} = \dfrac{dy}{dt}\dfrac{dv}{dy} = v\dfrac{dv}{dy}$.

Example: $\dfrac{d^2y}{dt^2} = -\dfrac{gR^2}{y^2}$.

After substitution, the equation becomes $v\dfrac{dv}{dy} + \dfrac{gR^2}{y^2} = 0$, which is separable.

VII. Duh.

If you find a solution to a differential equation using one of the methods in these notes but cannot fix a constant so that the initial condition is satisfied, first check to see if the initial condition is even possible to satisfy (by sticking the initial time into the equation). If it is, there is a good chance that your method involved integrating over some interval in which a function blows up. In this case, you try to use pure thought to eyeball a solution. A good thing to try is $y(t) = y_0$, a constant function.

Existence and uniqueness.

There are many versions of existence and uniqueness theorems for differential equations. Here is one.

Theorem. *Let $f(t, y)$ be a continuous function on a rectangle R in the ty-plane. Given a point $(t_0, y_0) \in R$, the initial value problem*

$$y' = f(t, y), \quad y(t_0) = y_0$$

has a solution $y(t)$ defined in an open interval containing t_0. The solution is defined for at least until the curve $t \mapsto (t, y(t))$ leaves R. Further, if $\partial f/\partial y$ exists and is continuous on R, then the solution is unique: if y_1 and y_2 are solutions then as long as $(t, y_1(t))$ and $(t, y_2(t))$ stay in R, we have $y_1(t) = y_2(t)$.

Example: Consider the differential equation

$$ty' = 2y - t^3y^2, \quad y(t_0) = y_0. \tag{1}$$

To apply the existence/uniqueness theorem we need to put the equation in standard form:

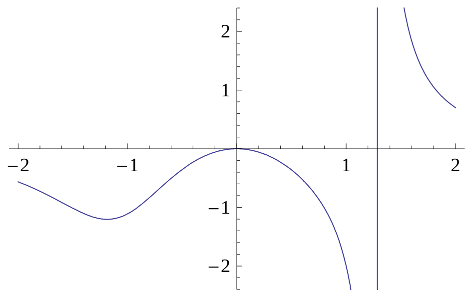
$$y' = \frac{2}{t}y - t^2y^2, \quad y(t_0) = y_0.$$

The theorem applies on any rectangle where the function $2y/t - t^2y^2$ is continuous, hence on any rectangle not containing a point with t-coordinate equal to 0. In fact, assuming $t_0 \neq 0$, the equation is a Bernoulli-type first-order linear equation with solution

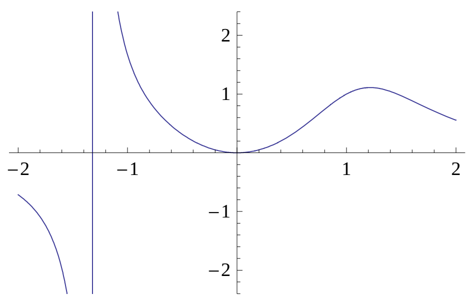
$$y(t) = \frac{5t^2}{t^5 + 5c}, \quad c = \frac{t_0^2}{y_0} - \frac{1}{5}t_0^5$$

provided $y_0 \neq 0$. If $y_0 = 0$, the solution is $y = 0$. Below are graphs of the solutions, $(t, y(t)$ for various initial conditions.

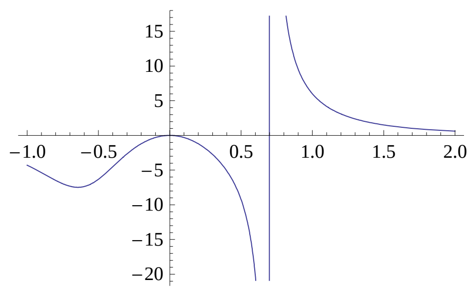
$y(1) = -2$:



$y(1) = 1$:



$y(1) = 6$:



Our existence/uniqueness theorem says nothing about solutions to problem 1 in the case where $t = 0$. In fact, letting $t = 0$ in the equation forces $y = 0$. Thus, for instance, there are no solutions satisfying the initial condition $y(0) = 1$. On the other hand, there are infinitely many solutions satisfying the initial condition $y(0) = 0$. All of the solutions given above have that property.