Laplace Transform and Systems of Ordinary Differential Equations

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Laplace Transform and Systems of Ordinary Differential Equations

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In this chapter, we describe a fundamental study of the Laplace transform, its use in the solution of initial value problems and some techniques to solve systems of ordinary differential equations (DE) including their solution with the help of the Laplace transform.

1. Laplace Transform

Improper Integrals. If $t_0 \in \mathbb{R}$, we define for any function $f(t)$

$$\int_{t_0}^{\infty} f(t)dt = \lim_{A\to\infty}\int_{t_0}^{A} f(t)dt$$

If this limit exists, we say that the improper integral converges, otherwise, diverges.

Example 1.

$$\int_1^{\infty}\frac{dx}{x^5} = \lim_{A\to\infty}\int_1^{A} x^{-5}\,dx = \lim_{A\to\infty} -\frac{1}{4x^4}\Big|_1^A = \lim_{A\to\infty} -\frac{1}{4}\left(\frac{1}{A^4} - \frac{1}{1^4}\right) = \frac{1}{4}$$

Initial Function and its Transform. Let f be a function of a single real variable t defined for any $t \geq 0$ and let e^{-st} be a complex function of a real variable t, where $s = a + bi$, $a > 0$. Let us examine the product

$$e^{-st}f(t), \qquad (1)$$

where (1) is also a complex function of a real variable t:

$$\begin{aligned} e^{-st}f(t) &= f(t)e^{-st} = f(t)e^{-(a+bi)t} = f(t)e^{-at}e^{-ibt} = f(t)e^{-at}(\cos bt - i\sin bt) \\ &= f(t)e^{-at}\cos bt - i\,f(t)e^{-at}\sin bt. \qquad (2) \end{aligned}$$

From (2), we get

$$\int_0^{\infty} e^{-st}f(t)dt = \int_0^{\infty} f(t)e^{-at}\cos bt\,dt - i\int_0^{\infty} f(t)e^{-at}\sin bt\,dt. \qquad (3)$$

The left side of (3) determines a function of s, which is called $F(s)$. So

$$F(s) = \int_0^{\infty} e^{-st}f(t)\,dt. \qquad (4)$$

Function (4) is called the *Laplace transform* or briefly, *$\mathcal{L}$-transform*, and function $f(t)$ is called its *initial function*. If $F(s)$ is the $\mathcal{L}$-transform of function $f(t)$, then we write

$$\mathcal{L}\{f(t)\} = F(s). \qquad (5)$$

A function f is said to be of *exponential order* on the interval $[0, +\infty)$ if there exist constants C and α such that

$$|f(t)| \leq Ce^{\alpha t}. \qquad (6)$$

Theorem 1. If, according to (6), f is of exponential order and $a > \alpha$, then the Laplace transform

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

does exist.

Proof. This theorem will be true if we are able to prove that each of the integrals on the right side of (3) exists. Let us evaluate the first of these two integrals:

$$\begin{aligned}\int_0^\infty f(t)e^{-at}\cos bt\, dt &\leq \left|\int_0^\infty f(t)e^{-at}\cos bt\, dt\right| \\ &\leq \int_0^\infty |f(t)e^{-at}\cos bt| dt \leq \int_0^\infty |f(t)e^{-at}| dt \\ &\leq \int_0^\infty Ce^{\alpha t}e^{-at} dt \\ &= C\int_0^\infty e^{(\alpha-a)t} dt = C\lim_{A\to\infty}\int_0^A e^{(\alpha-a)t} dt = C\lim_{A\to\infty}\frac{1}{\alpha-a}\, e^{(\alpha-a)t}\Big|_0^A \\ &= C\lim_{A\to\infty}\frac{1}{\alpha-a}\left[e^{(\alpha-a)A}-1\right] = \frac{c}{\alpha-a}\lim_{A\to\infty}\left[e^{-(a-\alpha)A}-1\right] = \frac{C}{a-\alpha}.\end{aligned}$$

Therefore

$$\int_0^\infty f(t)e^{-at}\cos bt\, dt \leq \frac{C}{a-\alpha}. \qquad (7)$$

Let us evaluate the second integral:

$$\begin{aligned}\int_0^\infty f(t)e^{-at}\sin bt\, dt \leq \left|\int_0^\infty f(t)e^{-at}\sin bt\, dt\right| \leq \int_0^\infty |f(t)e^{-at}\sin bt| dt \leq \int_0^\infty |f(t)e^{-at}| dt \\ \leq \int_0^\infty Ce^{\alpha t}e^{-at} dt = \frac{C}{a-\alpha}. \qquad (8)\end{aligned}$$

Thus, both integrals (7) and (8) do exist. We can then conclude that

$$\int_0^\infty e^{-st} f(t) dt$$

also exists and so Theorem 1 is proved. □

$\mathcal{L}$-transforms of functions $\sin t$ and $\cos t$

I. If $f(t) = \sin t$, then

$$\mathcal{L}\{\sin t\} = \int_0^\infty e^{-st} \sin t \, dt = \lim_{A\to\infty} \int_0^A e^{-st} \sin t \, dt = \lim_{A\to\infty} \frac{1}{1+s^2}\left(1 - \frac{\cos A}{e^{As}} - \frac{s \sin A}{e^{As}}\right)$$
$$= \frac{1}{s^2+1} \ . \qquad (9)$$

II. If $f(t) = \cos t$, then

$$\mathcal{L}\{\cos t\} = \int_0^\infty e^{-st} \cos t \ dt = \lim_{A\to\infty} \int_0^A e^{-st} \cos t \ dt = \lim_{A\to\infty} \frac{1}{1+s^2}\left(s + \frac{\sin A}{e^{As}} - \frac{s \cos A}{e^{As}}\right)$$
$$= \frac{s}{s^2+1} \ . \qquad (10)$$

Change of Scale Property

Theorem 2. The $\mathcal{L}$-transform of $f(at)$, $a > 0$, is

$$\mathcal{L}\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right). \qquad (11)$$

Proof. We know that

$$\mathcal{L}\{f(at)\} = \int_0^\infty e^{-st} f(at) dt.$$

Making $z = at \ \Rightarrow \ \frac{1}{a} dz = dt$, then

$$\mathcal{L}\{f(at)\} = \int_0^\infty e^{-\frac{sz}{a}} f(z) \frac{dz}{a} = \frac{1}{a}\int_0^\infty e^{-\frac{s}{a}z} f(z) dz = \frac{1}{a} F\left(\frac{s}{a}\right). \square$$

Example 2. Find the $\mathcal{L}$-transform of $f(at) = \sin at$.

If $f(z) = \sin z$, then, according to (9),

$$F\left(\frac{s}{a}\right) = \int_0^\infty e^{-\frac{s}{a}z} \sin z \, dz = \frac{1}{\left(\frac{s}{a}\right)^2 + 1} = \frac{a^2}{s^2+a^2}$$

and according to (11), the $\mathcal{L}$-transform of $f(at) = \sin at$ is

$$\mathcal{L}\{\sin at\} = \frac{1}{a} F\left(\frac{s}{a}\right) = \frac{1}{a} \frac{a^2}{s^2+a^2} = \frac{a}{s^2+a^2} \ . \qquad (12)$$

Example 3. Find the $\mathcal{L}$-transform of $f(at) = \cos at$.

If $f(z) = \cos z$, then, according to (10),

$$F\left(\frac{s}{a}\right) = \int_0^\infty e^{-\frac{s}{a}z}\cos z\ dz = \frac{\frac{s}{a}}{\left(\frac{s}{a}\right)^2 + 1} = \frac{as}{s^2 + a^2}$$

and according to (11), the $\mathcal{L}$-transform of $f(at) = \cos at$ is

$$\mathcal{L}\{\cos at\} = \frac{1}{a}F\left(\frac{s}{a}\right) = \frac{1}{a}\frac{as}{s^2 + a^2} = \frac{s}{s^2 + a^2}\,. \qquad (13)$$

$\mathcal{L}$-transforms of functions $f(t) = 1, f(t) = t^n$

I. If $f(t) = 1$, then

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}(1)dt = \int_0^\infty e^{-st}dt = \lim_{A\to\infty}\int_0^A e^{-st}dt = \lim_{A\to\infty} -\frac{1}{s}e^{-st}\Big|_0^A = \lim_{A\to\infty} -\frac{1}{s}(e^{-At} - e^0)$$
$$= \lim_{A\to\infty}\frac{1}{s}\left(1 - \frac{1}{e^{At}}\right) = \frac{1}{s}\,. \qquad (14)$$

II. If $f(t) = t^n$, then, by using integration by parts successively, we have

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st}t^n dt = \lim_{A\to\infty}\int_0^A e^{-st}t^n\,dt = \lim_{A\to\infty}\frac{n!}{s^n}\int_0^A e^{-st}dt = \lim_{A\to\infty}\frac{n!}{s^n}\left(-\frac{1}{s}e^{-st}\right)\Big|_0^A$$
$$= \lim_{A\to\infty} -\frac{n!}{s^n(s)}(e^{-As} - e^0) = \frac{n!}{s^{n+1}}\lim_{A\to\infty}\left(1 - \frac{1}{e^{As}}\right) = \frac{n!}{s^{n+1}}\,. \qquad (15)$$

Linearity Property

Theorem 3. If a and b are constants, then

$$\mathcal{L}\{af + bg\} = a\mathcal{L}\{f\} + b\mathcal{L}\{g\}. \qquad (16)$$

Proof. Applying the definition of the Laplace transform, we get

$$\mathcal{L}\{af + bg\} = \int_0^\infty e^{-st}(af + bg)dt = \int_0^\infty e^{-st}afdt + \int_0^\infty e^{-st}bgdt = a\int_0^\infty e^{-st}fdt + b\int_0^\infty e^{-st}gdt$$
$$= a\mathcal{L}\{f\} + b\mathcal{L}\{g\}. \ \square$$

Example 4. Find the $\mathcal{L}$-transform of $f(t) = t^2 + 8t - 16$.

$$\mathcal{L}\{t^2 + 8t - 16\} = \mathcal{L}\{t^2\} + 8\mathcal{L}\{t\} - 16\mathcal{L}\{1\} = \frac{2!}{s^{2+1}} + 8\left(\frac{1!}{s^{1+1}}\right) - 16\left(\frac{1}{s}\right) = \frac{2}{s^3} + \frac{8}{s^2} - \frac{16}{s}\,.$$

Shift Theorem (or Shifting Property)

Theorem 4. If $F(s)$ is the $\mathcal{L}$-transform of $f(t)$, then $F(s + \alpha)$ is the $\mathcal{L}$-transform of $e^{-\alpha t}f(t)$, that is,

if $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\{e^{-\alpha t}f(t)\} = F(s + \alpha). \qquad (17)$$

Proof.

$$\boldsymbol{\mathcal{L}}\{e^{-\alpha t}f(t)\} = \int_0^\infty e^{-st}e^{-\alpha t}f(t)dt = \int_0^\infty e^{-(s+\alpha)t}f(t)dt = F(s+\alpha). \ \square$$

$\boldsymbol{\mathcal{L}}$-transforms of functions $\boldsymbol{e^{-\alpha t}, e^{-\alpha t}\sin at,\ e^{-\alpha t}\cos at}$

I. Find the $\mathcal{L}$-transform of $e^{-\alpha t}$.

From (17), making *f* (*t*) = 1 and keeping in mind that $\mathcal{L}\{f(t)\} = \mathcal{L}\{1\} = \frac{1}{s} = F(s)$, we get

$$\mathcal{L}\{e^{-\alpha t}\} = \mathcal{L}\{e^{-\alpha t}(1)\} = \mathcal{L}\{e^{-\alpha t}f(t)\} = F(s+\alpha) = \frac{1}{s+\alpha}. \qquad (18)$$

II. Find the $\mathcal{L}$-transform of $e^{-\alpha t}\sin at$.

From (17), making *f* (*t*) = sin *at* and keeping in mind that $\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2} = F(s)$, we get

$$\mathcal{L}\{e^{-\alpha t}\sin at\} = \mathcal{L}\{e^{-\alpha t}f(t)\} = F(s+\alpha) = \frac{a}{(s+\alpha)^2+a^2}. \qquad (19)$$

III. Find the $\mathcal{L}$-transform of $e^{-\alpha t}\cos at$.

From (17), making *f* (*t*) = cos *at* and keeping in mind that $\mathcal{L}\{f(t)\} = \mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2} = F(s)$, we get

$$\mathcal{L}\{e^{-\alpha t}\cos at\} = \mathcal{L}\{e^{-\alpha t}f(t)\} = F(s+\alpha) = \frac{s+\alpha}{(s+\alpha)^2+a^2}. \qquad (20)$$

N-th Order Derivative Property

Theorem 5. If *f* (*t*) is of exponential order and *F*(*s*) is the $\mathcal{L}$-transform of *f*, then

$$(-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f(t)\} = \mathcal{L}\{t^n f(t)\}. \qquad (21)$$

Proof. Take the derivative of both sides of the equation

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt$$

n times with respect to *s*.

i)

$$\frac{d}{ds}\mathcal{L}\{f(t)\} = \frac{d}{ds}\int_0^\infty e^{-st}f(t)\,dt = \int_0^\infty \frac{\partial}{\partial s}[e^{-st}f(t)]dt = -\int_0^\infty e^{-st}tf(t)dt,$$

which means,

$$-\frac{d}{ds}\mathcal{L}\{f(t)\} = \mathcal{L}\{tf(t)\}$$

ii)

$$\frac{d^2}{ds^2}\mathcal{L}\{f(t)\} = -\int_0^\infty \frac{\partial}{\partial s}[e^{-st}tf(t)]dt = \int_0^\infty e^{-st}t^2 f(t)\,dt = \mathcal{L}\{t^2 f(t)\}$$

iii)

$$\frac{d^3}{ds^3}\mathcal{L}\{f(t)\} = \int_0^\infty \frac{\partial}{\partial s}[e^{-st}t^2 f(t)]dt = -\int_0^\infty e^{-st}t^3 f(t)\,dt,$$

which means,

$$-\frac{d^3}{ds^3}\mathcal{L}\{f(t)\} = \mathcal{L}\{t^3 f(t)\}$$

and so on. Thus, it follows that

$$(-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f(t)\} = \mathcal{L}\{t^n f(t)\}. \ \square$$

$\mathcal{L}$-transforms of functions $t\sin at$ and $t\cos at$

I. Find the $\mathcal{L}$-transform of $t\sin at$.

From (21), making $f(t) = \sin at$ and keeping in mind (12), we obtain

$$\mathcal{L}\{t\sin at\} = \mathcal{L}\{tf(t)\} = -\frac{d}{ds}\mathcal{L}\{f(t)\} = -\frac{d}{ds}\mathcal{L}\{\sin at\} = -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right) = -\left[-\frac{2as}{(s^2+a^2)^2}\right]$$
$$= \frac{2as}{(s^2+a^2)^2}. \qquad (22)$$

II. Find the $\mathcal{L}$-transform of $t\cos at$.

From (21), making $f(t) = \cos at$ and keeping in mind (13), we obtain

$$\mathcal{L}\{t\cos at\} = \mathcal{L}\{tf(t)\} = -\frac{d}{ds}\mathcal{L}\{f(t)\} = -\frac{d}{ds}\mathcal{L}\{\cos at\} = -\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right) = -\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$
$$= \frac{s^2-a^2}{(s^2+a^2)^2}. \qquad (23)$$

Laplace Transform of derivatives

Theorem 6. If $f(t)$ is a function of exponential order, then

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0). \qquad (24)$$

Proof.

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)dt$$
$$= \lim_{A\to\infty} \int_0^A e^{-st} f'(t)dt$$
$$= \lim_{A\to\infty} \left[e^{-st} f(t) \Big|_0^A + \int_0^A s e^{-st} f(t) dt \right]$$
$$= \lim_{A\to\infty} \left[\frac{f(A)}{e^{As}} - f(0) \right] + s \lim_{A\to\infty} \int_0^A e^{-st} f(t)dt = s\int_0^\infty e^{-st} f(t)dt - f(0)$$
$$= s\mathcal{L}\{f(t)\} - f(0). \ \square$$

Remark 1.

a)

$$\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0) = s[s\mathcal{L}\{f(t)\} - f(0)] - f'(0) = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0).$$

b)

$$\mathcal{L}\{f'''(t)\} = s\mathcal{L}\{f''(t)\} - f''(0) = s[s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)] - f''(0)$$
$$= s^3\mathcal{L}\{f(t)\} - s^2 f(0) - sf'(0) - f''(0).$$

c)

$$\mathcal{L}\left\{f^{(4)}(t)\right\} = s\mathcal{L}\{f'''(t)\} - f'''(0) = s[s^3\mathcal{L}\{f(t)\} - s^2 f(0) - sf'(0) - f''(0)] - f'''(0)$$
$$= s^4\mathcal{L}\{f(t)\} - s^3 f(0) - s^2 f'(0) - sf''(0) - f'''(0).$$

And so on. In general,

d)

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{(n-1)}(0). \qquad (25)$$

Inverse Laplace Transform

The original function $f(t)$ is called the *inverse* $\mathcal{L}$*-transform* of $F(s) = \mathcal{L}\{f(t)\}$ and it is designated by

$$f(t) = \mathcal{L}^{-1}\left\{\mathcal{L}\{f(t)\}\right\}$$

Example 5. Find $\mathcal{L}^{-1}\left\{\mathcal{L}\{f(t)\}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s^2}\right\}$

Keeping in mind (15), we obtain

$$f(t) = \mathcal{L}^{-1}\left\{\frac{2}{s^2}\right\} = \mathcal{L}^{-1}\left\{2\left(\frac{1!}{s^{1+1}}\right)\right\} = 2\,\mathcal{L}^{-1}\left\{\frac{1!}{s^{1+1}}\right\} = 2t^1 = 2t$$

Table 1. Some $\mathcal{L}$-transforms

#	$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$	$f(t)$
1.	$\frac{1}{s}$	1
2.	$\frac{a}{s^2 + a^2}$	$\sin at$
3.	$\frac{s}{s^2 + a^2}$	$\cos at$
4.	$\frac{2as}{(s^2 + a^2)^2}$	$t \sin at$
5.	$\frac{s^2 - a^2}{(s^2 + a^2)^2}$	$t \cos at$
6.	$\frac{1}{s \mp \alpha}$	$e^{\pm \alpha t}$
7.	$\frac{1}{(s \mp \alpha)^2}$	$te^{\pm \alpha t}$
8.	$\frac{a}{(s \mp \alpha)^2 + a^2}$	$e^{\pm \alpha t} \sin at$
9.	$\frac{s \mp \alpha}{(s \mp \alpha)^2 + a^2}$	$e^{\pm \alpha t} \cos at$
10.	$\frac{a}{(s \mp \alpha)^2 - a^2}$	$e^{\pm \alpha t} \sinh at$
11.	$\frac{s \mp \alpha}{(s \mp \alpha)^2 - a^2}$	$e^{\pm \alpha t} \cosh at$
12.	$\frac{n!}{s^{n+1}}$	t^n
13.	$(-1)^n \frac{d^n}{ds^n} F(s)$	$t^n f(t)$

Example 6. Find $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$.

Keeping in mind (12), we get

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{2}\left(\frac{2}{s^2+2^2}\right)\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\} = \frac{1}{2}\sin 2t$$

Solution of Initial Value Problems Using the Laplace Transform

Example 7. Solve the initial value problem

$$y'' + 2y' + 5y = 0, \qquad y(0) = y'(0) = 1.$$

Let us apply the Laplace transform to both sides of the given DE

$$\mathcal{L}\{y'' + 2y' + 5y\} = \mathcal{L}\{0\}$$

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = 0$$

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 2[s\mathcal{L}\{y\} - y(0)] + 5\mathcal{L}\{y\} = 0$$

$$s^2\mathcal{L}\{y\} - s - 1 + 2s\mathcal{L}\{y\} - 2 + 5\mathcal{L}\{y\} = 0$$

$$(s^2 + 2s + 5)\mathcal{L}\{y\} = s + 3$$

$$\mathcal{L}\{y\} = \frac{s+3}{s^2+2s+5}$$

$$\mathcal{L}\{y\} = \frac{s+3}{(s+1)^2+4} = \frac{s+1}{(s+1)^2+2^2} + \frac{2}{(s+1)^2+2^2}$$

$$y = \mathcal{L}^{-1}\{\mathcal{L}\{y\}\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+2^2}\right\}$$

$$y = e^{-t}\cos 2t + e^{-t}\sin 2t$$

$$y = e^{-t}(\cos 2t + \sin 2t)$$

Example 8. Solve the initial value problem

$$y'' - 2y' - 3y = 5, \qquad y(0) = 0,\ y'(0) = 1.$$

Let us apply the Laplace transform to both sides of the given DE

$$\mathcal{L}\{y'' - 2y' - 3y\} = \mathcal{L}\{5\}$$

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} - 3\mathcal{L}\{y\} = \mathcal{L}\{5(1)\}$$

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) - 2[s\mathcal{L}\{y\} - y(0)] - 3\mathcal{L}\{y\} = 5\mathcal{L}\{1\}$$

$$s^2\mathcal{L}\{y\} - 1 - 2s\mathcal{L}\{y\} - 3\mathcal{L}\{y\} = 5\left(\frac{1}{s}\right)$$

$$(s^2 - 2s - 3)\mathcal{L}\{y\} = \frac{5}{s} + 1$$

$$\mathcal{L}\{y\} = \frac{s+5}{s(s^2 - 2s - 3)}.$$

We need to use the partial fraction expansion of a rational expression

$$\frac{s+5}{s(s^2 - 2s - 3)} = \frac{A}{s} + \frac{Bs + C}{s^2 - 2s - 3}$$

$$s + 5 = A(s^2 - 2s - 3) + (Bs + C)s$$

$$s + 5 = (A + B)s^2 + (-2A + C)s - 3A.$$

Let us solve the system of linear equations

$$\left.\begin{array}{rl} A + B \qquad & = 0 \\ -2A \qquad + C & = 1 \\ -3A \qquad\qquad & = 5 \end{array}\right\}$$

whose solution is

$$A = -\frac{5}{3}, B = \frac{5}{3}, C = -\frac{7}{3},$$

then

$$\mathcal{L}\{y\} = \frac{s+5}{s(s^2 - 2s - 3)} = \frac{1}{3}\left(-\frac{5}{s} + \frac{5s - 7}{s^2 - 2s - 3}\right)$$

$$y = \mathcal{L}^{-1}\{\mathcal{L}\{y\}\} = \frac{1}{3}\left[-5\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{5s-5}{(s-1)^2 - 2^2} - \frac{2}{(s-1)^2 - 2^2}\right\}\right]$$

$$y = \frac{1}{3}\left[-5 + 5\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2 - 2^2}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2 - 2^2}\right\}\right]$$

$$y = \frac{1}{3}(-5 + 5e^t \cosh 2t - e^t \sinh 2t)$$

2. Systems of Ordinary DE with Constant Coefficients

Given the set of numbers a_{ij} and functions $f_i(t)$, the expression

$$\frac{dy_i}{dt} = \sum_{j=1}^{n} a_{ij} y_j + f_i(t) \quad (i = 1,2, \dots, n) \qquad (1)$$

which is equivalent to

$$
\begin{aligned}
\frac{dy_1}{dt} &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n + f_1(t) \\
\frac{dy_2}{dt} &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n + f_2(t) \\
&\ldots \qquad\qquad\qquad\qquad\qquad\qquad (1A) \\
\frac{dy_n}{dt} &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n + f_n(t)
\end{aligned}
$$

is called a *system of linear nonhomogeneous DE with constant coefficients*.

On the other hand, (1*A*) is called *a system of linear homogeneous DE with constant coefficients* if all of the functions $f_i(t)$ are zero.

If the set of functions

$$
\begin{aligned}
y_1 &= \varphi_1(t) \\
y_2 &= \varphi_2(t) \\
&\ldots \qquad\qquad\qquad (2) \\
y_n &= \varphi_n(t)
\end{aligned}
$$

with continuous derivatives on the interval (a, b) convert the equations of system (1A) into identities for each value of $t \in (a, b)$, then (2) is called a solution of system (1A).

The simplest technique to solve system (1A) consists in reducing it to an n-th order DE. Let us illustrate this method in the example of a system of two DE

$$
\left.
\begin{aligned}
\frac{dy_1}{dt} &= ay_1 + by_2 + f(t) \qquad (3A) \\
\frac{dy_2}{dt} &= cy_1 + dy_2 + g(t) \qquad (3B)
\end{aligned}
\right\} \qquad (3)
$$

where a, b, c, d are constant coefficients, $f(t)$, $g(t)$ are given functions and $y_1(t)$, $y_2(t)$ are the unknown functions. From (3*A*), we get

$$by_2 = \frac{dy_1}{dt} - ay_1 - f(t)$$

$$y_2 = \frac{1}{b}\left[\frac{dy_1}{dt} - ay_1 - f(t)\right] \qquad (4)$$

Thus, we obtain from (4)

$$\frac{dy_2}{dt} = \frac{1}{b}\left[\frac{d^2y_1}{dt^2} - a\frac{dy_1}{dt} - f'(t)\right] \qquad (5)$$

Substituting (4) into (3*B*), we get

$$\frac{dy_2}{dt} = cy_1 + \frac{d}{b}\left[\frac{dy_1}{dt} - ay_1 - f(t)\right] + g(t) \qquad (6)$$

Equating (5) and (6), we obtain

$$\frac{1}{b}\left[\frac{d^2y_1}{dt^2} - a\frac{dy_1}{dt} - f'(t)\right] = cy_1 + \frac{d}{b}\left[\frac{dy_1}{dt} - ay_1 - f(t)\right] + g(t)$$

$$\frac{1}{b}\frac{d^2y_1}{dt^2} - \left(\frac{a+d}{b}\right)\frac{dy_1}{dt} + \left(\frac{ad}{b} - c\right)y_1 + \left[-\frac{1}{b}f'(t) + \frac{d}{b}f(t) - g(t)\right] = 0$$

and so, we finally get

$$A\frac{d^2y_1}{dt^2} + B\frac{dy_1}{dt} + Cy_1 + P(t) = 0$$

where *A*, *B*, *C* are constants. From the last DE, we find $y_1 = y_1(t,\ C_1,\ C_2)$, and substituting y_1 and dy_1/dt into (4), we get y_2.

Example 1. Find the general solution, or briefly, *integrate* the following system

$$\left.\begin{aligned} \frac{dy_1}{dt} &= 4y_1 - y_2 + (t+1) \qquad (7A) \\ \frac{dy_2}{dt} &= 2y_1 + y_2 + (t-1) \qquad (7B) \end{aligned}\right\} \qquad (7)$$

From (7*A*), we get

$$y_2 = -\frac{dy_1}{dt} + 4y_1 + (t+1) \qquad (8)$$

$$\frac{dy_2}{dt} = -\frac{d^2y_1}{dt^2} + 4\frac{dy_1}{dt} + 1. \qquad (9)$$

Substituting (8) into (7*B*), we obtain

$$\frac{dy_2}{dt} = 2y_1 + \left[-\frac{dy_1}{dt} + 4y_1 + (t+1)\right] + (t-1)$$

$$\frac{dy_2}{dt} = 6y_1 - \frac{dy_1}{dt} + 2t. \qquad (10)$$

By equating (9) and (10), we have

$$-\frac{d^2y_1}{dt^2} + 4\frac{dy_1}{dt} + 1 = 6y_1 - \frac{dy_1}{dt} + 2t$$

$$y_1'' - 5y_1' + 6y_1 = 1 - 2t. \qquad (11)$$

Let us find the general solution of (11). See article [1]. But first, let us find the solution of the corresponding linear homogeneous DE.

$$D^2y_1 - 5Dy_1 + 6y_1 = 0$$

$$(D^2 - 5D + 6)y_1 = 0$$

$$r^2 - 5r + 6 = 0 \qquad (12)$$

$$(r-2)(r-3) = 0$$

$$r_1 = 2,\ r_2 = 3.$$

Therefore, the solution of the corresponding linear homogeneous DE is

$$y_{1h} = C_1e^{2t} + C_2e^{3t} \qquad (13)$$

Now, let us find a particular solution of the linear nonhomogeneous DE (11) by using the method of undetermined coefficients. See [1]. $\alpha = 0$ is not a root of the characteristic equation (12). Therefore,

$$y_{1p} = e^{0x}\sum_{k=0}^{1} A_k t^k = A_0 + A_1 t. \qquad (14)$$

From (14), we get

$$y'_{1p} = A_1,\ y''_{1p} = 0.$$

From (11), we obtain

$$0 - 5A_1 + 6(A_0 + A_1 t) = 1 - 2t$$

$$6A_0 - 5A_1 + 6A_1 t = 1 - 2t$$

$$\left.\begin{aligned} 6A_0 - 5A_1 &= 1 \\ 6A_1 &= -2 \end{aligned}\right\}$$

$$A_0 = -\frac{1}{9}, \qquad A_1 = -\frac{1}{3}.$$

By substituting the values of A_0 and A_1 into (14), we have

$$y_{1p} = -\frac{t}{3} - \frac{1}{9}$$

Consequently

$$y_1 = y_{1h} + y_{1p} = C_1e^{2t} + C_2e^{3t} - \frac{t}{3} - \frac{1}{9}. \qquad (15)$$

Let us get the derivative of (15)

$$\frac{dy_1}{dt} = 2C_1e^{2t} + 3C_2e^{3t} - \frac{1}{3}\,. \qquad (16)$$

By substituting (15) and (16) into (8), we have

$$y_2 = -\left(2C_1e^{2t} + 3C_2e^{3t} - \frac{1}{3}\right) + 4\left(C_1e^{2t} + C_2e^{3t} - \frac{t}{3} - \frac{1}{9}\right) + (t+1)$$

$$y_2 = 2C_1e^{2t} + C_2e^{3t} - \frac{t}{3} + \frac{8}{9}.$$

Euler's Method for Integration of a System of Three Linear Homogeneous DE with Constant Coefficients

Let us consider the system

$$\begin{aligned} \frac{dy_1}{dt} &= ay_1 + by_2 + cy_3 \\ \frac{dy_2}{dt} &= a_1y_1 + b_1y_2 + c_1y_3 \\ \frac{dy_3}{dt} &= a_2y_1 + b_2y_2 + c_2y_3 \end{aligned} \qquad (17)$$

Let us assume that the solution of system (17) can be written in the form

$$y_1 = \lambda e^{rt}, \qquad y_2 = \mu e^{rt}, \qquad y_3 = \nu e^{rt} \qquad (18)$$

where λ, μ, ν, r are constants. From (18), we get

$$\frac{dy_1}{dt} = \lambda re^{rt}, \qquad \frac{dy_2}{dt} = \mu re^{rt}, \qquad \frac{dy_3}{dt} = \nu re^{rt}. \qquad (19)$$

By substituting (18), (19) into (17), we have

$$\begin{aligned} \lambda re^{rt} &= a\lambda e^{rt} + b\mu e^{rt} + c\nu e^{rt} \\ \mu re^{rt} &= a_1\lambda e^{rt} + b_1\mu e^{rt} + c_1\nu e^{rt} \\ \nu re^{rt} &= a_2\lambda e^{rt} + b_2\mu re^{rt} + c_2\nu e^{rt} \end{aligned}$$

which is equivalent to have

$$\left.\begin{aligned} (a-r)\lambda + b\mu + c\nu = 0 \\ a_1\lambda + (b_1 - r)\mu + c_1\nu = 0 \\ a_2\lambda + b_2\mu + (c_2 - r)\nu = 0 \end{aligned}\right\} \qquad (20)$$

System (20) has a nonzero solution only if its determinant Δ is zero, that is,

$$\Delta = \begin{vmatrix} a-r & b & c \\ a_1 & b_1 - r & c_1 \\ a_2 & b_2 & c_2 - r \end{vmatrix} = 0 \qquad (21)$$

(21) is called the *characteristic equation* associated with system (17).

Case I. Assume that the roots r_1, r_2, r_3 of the characteristic equation (21) are real and distinct. Replacing in (20) r by r_1 and solving system (20), we obtain numbers λ_1, μ_1, ν_1. The same applies for $r = r_2$ obtaining λ_2, μ_2, ν_2 and also for $r = r_3$ obtaining now λ_3, μ_3, ν_3. Finally, for the three collections of numbers λ, μ, ν, we have three systems of particular solutions given by

$$\left.\begin{array}{lll} y_1^{(1)} = \lambda_1 e^{r_1 t} & y_2^{(1)} = \mu_1 e^{r_1 t} & y_3^{(1)} = \nu_1 e^{r_1 t} \\ y_1^{(2)} = \lambda_2 e^{r_2 t} & y_2^{(2)} = \mu_2 e^{r_2 t} & y_3^{(2)} = \nu_2 e^{r_2 t} \\ y_1^{(3)} = \lambda_3 e^{r_3 t} & y_2^{(3)} = \mu_3 e^{r_3 t} & y_3^{(3)} = \nu_3 e^{r_3 t} \end{array}\right\} \qquad (22)$$

and the solution of (17) takes the form

$$\left.\begin{array}{l} y_1 = C_1 y_1^{(1)} + C_2 y_1^{(2)} + C_3 y_1^{(3)} \\ y_2 = C_1 y_2^{(1)} + C_2 y_2^{(2)} + C_3 y_2^{(3)} \\ y_3 = C_1 y_3^{(1)} + C_2 y_3^{(2)} + C_3 y_3^{(3)} \end{array}\right\} \qquad (23)$$

Example 2. Integrate the following system

$$\begin{aligned} \frac{dy_1}{dt} &= y_1 - y_2 + y_3 \\ \frac{dy_2}{dt} &= y_1 + y_2 - y_3 \\ \frac{dy_3}{dt} &= 2y_1 - y_2 \end{aligned}$$

Let us form the system of type (20)

$$\left.\begin{array}{l} (1-r)\lambda - \mu + \nu = 0 \\ \lambda + (1-r)\mu - \nu = 0 \\ 2\lambda \; - \; \mu \; - \; r\nu = 0 \end{array}\right\} \qquad (24)$$

and write the characteristic equation

$$\begin{aligned} \Delta = \begin{vmatrix} 1-r & -1 & 1 \\ 1 & 1-r & -1 \\ 2 & -1 & -r \end{vmatrix} &= (1-r)\begin{vmatrix} 1-r & -1 \\ -1 & -r \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & -r \end{vmatrix} + \begin{vmatrix} 1 & 1-r \\ 2 & -1 \end{vmatrix} \\ &= (1-r)[-r(1-r)-1] + (-r+2) + [-1-2(1-r)] \\ &= (1-r)(r^2-r-1) + (2-r) + (2r-3) = -r^3 + 2r^2 + r - 2 = 0 \\ &\Rightarrow r^3 - 2r^2 - r + 2 = 0. \end{aligned}$$

Therefore

$$(r-1)(r^2-r-2) = 0$$

$$(r-1)(r-2)(r+1) = 0$$

$$r_1 = 1, \qquad r_2 = 2, \qquad r_3 = -1.$$

Thus, from (24) and knowing that $r = r_1 = 1$, we get

$$
\begin{aligned}
-\mu_1 + \nu_1 &= 0\\
\lambda_1 \qquad - \nu_1 &= 0\\
2\lambda_1 - \mu_1 - \nu_1 &= 0
\end{aligned}
$$

By solving the last system of equations, we have

$$\begin{pmatrix}\lambda_1\\ \mu_1\\ \nu_1\end{pmatrix} = A\begin{pmatrix}1\\1\\1\end{pmatrix}, \qquad A = \nu_1,\ \ A \in \mathbb{R}.$$

If $A = 1$, then

$$\lambda_1 = 1, \qquad \mu_1 = 1, \qquad \nu_1 = 1$$

$$y_1^{(1)} = e^t, \quad y_2^{(1)} = e^t, \quad y_3^{(1)} = e^t$$

Now, from (24) and knowing that $r = r_2 = 2$, we get

$$
\begin{aligned}
-\lambda_2 - \mu_2 + \nu_2 &= 0\\
\lambda_2 - \mu_2 - \nu_2 &= 0\\
2\lambda_2 - \mu_2 - 2\nu_2 &= 0
\end{aligned}
$$

The solution of this system of equations is

$$\begin{pmatrix}\lambda_2\\ \mu_2\\ \nu_2\end{pmatrix} = A_1\begin{pmatrix}1\\0\\1\end{pmatrix}, \qquad A_1 = \nu_2,\ \ A_1 \in \mathbb{R}.$$

If $A_1 = 1$, then

$$\lambda_2 = 1, \qquad \mu_2 = 0, \qquad \nu_2 = 1$$

$$y_1^{(2)} = e^{2t}, \quad y_2^{(2)} = 0, \quad y_3^{(2)} = e^{2t}$$

Lastly, from (24) and knowing that $r = r_3 = -1$, we obtain

$$
\begin{aligned}
2\lambda_3 - \mu_3 + \nu_3 &= 0\\
\lambda_3 + 2\mu_3 - \nu_3 &= 0\\
2\lambda_3 - \mu_3 + \nu_3 &= 0
\end{aligned}
$$

The solution of this system of equations is

$$\begin{pmatrix}\lambda_3\\ \mu_3\\ \nu_3\end{pmatrix} = A_2\begin{pmatrix}-1/5\\3/5\\1\end{pmatrix}, \qquad A_2 = \nu_3,\ \ A_2 \in \mathbb{R}.$$

If $A_2 = 1$, then

$$\lambda_3 = -1/5, \qquad \mu_3 = 3/5, \qquad \nu_3 = 1$$

$$y_1^{(3)} = (-1/5)e^{-t}, \quad y_2^{(3)} = (3/5)e^{-t}, \quad y_3^{(3)} = e^{-t}$$

Consequently, the general solution of the given system is

$$\left.\begin{aligned} y_1 &= C_1 y_1^{(1)} + C_2 y_1^{(2)} + C_3 y_1^{(3)} = C_1 e^t + C_2 e^{2t} - (1/5) C_3 e^{-t} \\ y_2 &= C_1 y_2^{(1)} + C_2 y_2^{(2)} + C_3 y_2^{(3)} = C_1 e^t \qquad\qquad + (3/5) C_3 e^{-t} \\ y_3 &= C_1 y_3^{(1)} + C_2 y_3^{(2)} + C_3 y_3^{(3)} = C_1 e^t + C_2 e^{2t} + C_3 e^{-t} \end{aligned}\right\}$$

Case II. Assume that r_1 is a root of the characteristic equation and that $r_2 = r_3 = r$ is a double root of it. Replacing in (20) r by r_1 and solving system (20), we obtain numbers λ, μ, ν. Thus, the following particular solutions are obtained

$$y_1^{(1)} = C_1 \lambda e^{r_1 t} \qquad y_2^{(1)} = C_1 \mu e^{r_1 t} \qquad y_3^{(1)} = C_1 \nu e^{r_1 t}$$

On the other hand, since $r_2 = r_3 = r$ is a double root of the characteristic equation, then we can assign to r a system of particular solutions of the form

$$y_1^{(2)} = (\lambda_1 t + \mu_1) e^{rt} \qquad y_2^{(2)} = (\lambda_2 t + \mu_2) e^{rt} \qquad y_3^{(2)} = (\lambda_3 t + \mu_3) e^{rt} \qquad (25)$$

From (25), we obtain

$$\left.\begin{aligned} \frac{dy_1^{(2)}}{dt} &= r(\lambda_1 t + \mu_1) e^{rt} + \lambda_1 e^{rt} = [r(\lambda_1 t + \mu_1) + \lambda_1] e^{rt} \\ \frac{dy_2^{(2)}}{dt} &= r(\lambda_2 t + \mu_2) e^{rt} + \lambda_2 e^{rt} = [r(\lambda_2 t + \mu_2) + \lambda_2] e^{rt} \\ \frac{dy_3^{(2)}}{dt} &= r(\lambda_3 t + \mu_3) e^{rt} + \lambda_3 e^{rt} = [r(\lambda_3 t + \mu_3) + \lambda_3] e^{rt} \end{aligned}\right\} \qquad (26)$$

By substituting (25) and (26) into (17), we have

$$\begin{aligned} r(\lambda_1 t + \mu_1) + \lambda_1 &= a(\lambda_1 t + \mu_1) + b(\lambda_2 t + \mu_2) + c(\lambda_3 t + \mu_3) \\ r(\lambda_2 t + \mu_2) + \lambda_2 &= a_1(\lambda_1 t + \mu_1) + b_1(\lambda_2 t + \mu_2) + c_1(\lambda_3 t + \mu_3) \\ r(\lambda_3 t + \mu_3) + \lambda_3 &= a_2(\lambda_1 t + \mu_1) + b_2(\lambda_2 t + \mu_2) + c_2(\lambda_3 t + \mu_3) \end{aligned}$$

which is equivalent to

$$\left.\begin{aligned} r\lambda_1 t + (r\mu_1 + \lambda_1) &= (a\lambda_1 + b\lambda_2 + c\lambda_3)t + (a\mu_1 + b\mu_2 + c\mu_3) \\ r\lambda_2 t + (r\mu_2 + \lambda_2) &= (a_1\lambda_1 + b_1\lambda_2 + c_1\lambda_3)t + (a_1\mu_1 + b_1\mu_2 + c_1\mu_3) \\ r\lambda_3 t + (r\mu_3 + \lambda_3) &= (a_2\lambda_1 + b_2\lambda_2 + c_2\lambda_3)t + (a_2\mu_1 + b_2\mu_2 + c_3\mu_3) \end{aligned}\right\} \qquad (27)$$

From (27), we obtain two systems of equations; the first one is given by

$$\left.\begin{aligned} a\lambda_1 + b\lambda_2 + c\lambda_3 &= r\lambda_1 \\ a_1\lambda_1 + b_1\lambda_2 + c_1\lambda_3 &= r\lambda_2 \\ a_2\lambda_1 + b_2\lambda_2 + c_2\lambda_3 &= r\lambda_3 \end{aligned}\right\}$$

and the second one is given by

$$\left.\begin{aligned} a\mu_1 + b\mu_2 + c\mu_3 = r\mu_1 + \lambda_1 \\ a_1\mu_1 + b_1\mu_2 + c_1\mu_3 = r\mu_2 + \lambda_2 \\ a_2\mu_1 + b_2\mu_2 + c_2\mu_3 = r\mu_3 + \lambda_3 \end{aligned}\right\}$$

with which we can find the values of $\lambda_1,\ \lambda_2,\ \lambda_3,\ \mu_1,\ \mu_2,\ \mu_3.$

Lastly, the solution of the system is

$$\left.\begin{aligned} y_1 = y_1^{(1)} + y_1^{(2)} = C_1\lambda e^{r_1 t} + (\lambda_1 t + \mu_1)e^{rt} \\ y_2 = y_2^{(1)} + y_2^{(2)} = C_1\mu e^{r_1 t} + (\lambda_2 t + \mu_2)e^{rt} \\ y_3 = y_3^{(1)} + y_3^{(2)} = C_1\nu e^{r_1 t} + (\lambda_3 t + \mu_3)e^{rt} \end{aligned}\right\}$$

Example 3. Integrate the system

$$\begin{aligned} \frac{dy_1}{dt} &= 2y_1 + 2y_2 - y_3 \\ \frac{dy_2}{dt} &= -2y_1 + 4y_2 + y_3 \\ \frac{dy_3}{dt} &= -3y_1 + 8y_2 + 2y_3 \end{aligned}$$

The system of type (20) has the form

$$\left.\begin{aligned} (2-r)\lambda + 2\mu - \nu = 0 \\ -2\lambda + (4-r)\mu + \nu = 0 \\ -3\lambda + 8\mu + (2-r)\nu = 0 \end{aligned}\right\} \qquad (28)$$

and the corresponding characteristic equation can be written in the form

$$\begin{aligned} \Delta = \begin{vmatrix} 2-r & 2 & -1 \\ -2 & 4-r & 1 \\ -3 & 8 & 2-r \end{vmatrix} &= (2-r)\begin{vmatrix} 4-r & 1 \\ 8 & 2-r \end{vmatrix} - 2\begin{vmatrix} -2 & 1 \\ -3 & 2-r \end{vmatrix} - \begin{vmatrix} -2 & 4-r \\ -3 & 8 \end{vmatrix} \\ &= (2-r)[(4-r)(2-r)-8] - 2[-2(2-r)+3] - [-16+3(4-r)] \\ &= (2-r)(r^2-6r) - 2(2r-1) - (-3r-4) = -r^3 + 8r^2 - 13r + 6 = 0 \\ &\Rightarrow r^3 - 8r^2 + 13r - 6 = 0. \end{aligned}$$

Therefore

$$(r-1)(r^2-7r+6) = 0$$

$$(r-1)(r-6)(r-1) = 0$$

$$r_1 = 6, \qquad r_2 = r_3 = r = 1.$$

Thus, from (28) and knowing that $r = r_1 = 6$, we get

$$\left.\begin{aligned} -4\lambda + 2\mu - \nu = 0 \\ -2\lambda - 2\mu + \nu = 0 \\ -3\lambda + 8\mu - 4\nu = 0 \end{aligned}\right\}$$

whose solution is

$$\begin{pmatrix}\lambda\\ \mu\\ \nu\end{pmatrix} = A\begin{pmatrix}0\\ 1/2\\ 1\end{pmatrix}, \qquad A = \nu,\ \ A \in \mathbb{R}.$$

If $A = 2$, then

$$\lambda = 0, \qquad \mu = 1, \qquad \nu = 2$$

$$y_1^{(1)} = 0, \quad y_2^{(1)} = C_1 e^{6t}, \quad y_3^{(1)} = 2C_1 e^{6t}. \tag{29}$$

Since $r_2 = r_3 = r = 1$ is a double root of $\Delta = 0$, then

$$y_1^{(2)} = (\lambda_1 t + \mu_1)e^t \qquad y_2^{(2)} = (\lambda_2 t + \mu_2)e^t \qquad y_3^{(2)} = (\lambda_3 t + \mu_3)e^t. \tag{30}$$

It follows that

$$\left.\begin{aligned}\frac{dy_1^{(2)}}{dt} &= (\lambda_1 t + \mu_1)e^t + \lambda_1 e^t = [\lambda_1 t + (\lambda_1 + \mu_1)]e^t\\ \frac{dy_2^{(2)}}{dt} &= (\lambda_2 t + \mu_2)e^t + \lambda_2 e^t = [\lambda_2 t + (\lambda_2 + \mu_2)]e^t\\ \frac{dy_3^{(2)}}{dt} &= (\lambda_3 t + \mu_3)e^t + \lambda_3 e^t = [\lambda_3 t + (\lambda_3 + \mu_3)]e^t\end{aligned}\right\} \tag{31}$$

By substituting (30) and (31) into the original system, we have

$$\left.\begin{aligned}\lambda_1 t + (\lambda_1 + \mu_1) &= (2\lambda_1 + 2\lambda_2 - \lambda_3)t + (2\mu_1 + 2\mu_2 - \mu_3)\\ \lambda_2 t + (\lambda_2 + \mu_2) &= (-2\lambda_1 + 4\lambda_2 + \lambda_3)t + (-2\mu_1 + 4\mu_2 + \mu_3)\\ \lambda_3 t + (\lambda_3 + \mu_3) &= (-3\lambda_1 + 8\lambda_2 + 2\lambda_3)t + (-3\mu_1 + 8\mu_2 + 2\mu_3)\end{aligned}\right\}$$

We can now form two systems of equations. The first one is given by

$$\left.\begin{aligned}2\lambda_1 + 2\lambda_2 - \lambda_3 &= \lambda_1\\ -2\lambda_1 + 4\lambda_2 + \lambda_3 &= \lambda_2\\ -3\lambda_1 + 8\lambda_2 + 2\lambda_3 &= \lambda_3\end{aligned}\right\} \Longrightarrow \left.\begin{aligned}\lambda_1 + 2\lambda_2 - \lambda_3 &= 0\\ -2\lambda_1 + 3\lambda_2 + \lambda_3 &= 0\\ -3\lambda_1 + 8\lambda_2 + \lambda_3 &= 0\end{aligned}\right\}$$

whose solution is

$$\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix} = C\begin{pmatrix}5/7\\ 1/7\\ 1\end{pmatrix}, \qquad C = \lambda_3,\ \ C \in \mathbb{R}.$$

If we make $C = 7C_2$, then

$$\lambda_1 = 5C_2, \qquad \lambda_2 = C_2, \qquad \lambda_3 = 7C_2. \tag{32}$$

The second system is given by

$$\left.\begin{aligned} 2\mu_1 + 2\mu_2 - \mu_3 &= \lambda_1 + \mu_1 \\ -2\mu_1 + 4\mu_2 + \mu_3 &= \lambda_2 + \mu_2 \\ -3\mu_1 + 8\mu_2 + 2\mu_3 &= \lambda_3 + \mu_3 \end{aligned}\right\} \Longrightarrow \left.\begin{aligned} \mu_1 + 2\mu_2 - \mu_3 &= 5C_2 \\ -2\mu_1 + 3\mu_2 + \mu_3 &= C_2 \\ -3\mu_1 + 8\mu_2 + \mu_3 &= 7C_2 \end{aligned}\right\}$$

whose solution is

$$\begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = C_2 \begin{pmatrix} 13/7 \\ 11/7 \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} 5/7 \\ 1/7 \\ 1 \end{pmatrix}, \qquad C_3 = \mu_3, \;\; C_3 \in \mathbb{R}.$$

This gives

$$\mu_1 = \frac{1}{7}(13C_2 + 5C_3), \qquad \mu_2 = \frac{1}{7}(11C_2 + C_3), \qquad \mu_3 = C_3. \tag{33}$$

By substituting (32) and (33) into (30), we have

$$y_1^{(2)} = \left[5C_2 t + \frac{1}{7}(13C_2 + 5C_3)\right] e^t, \quad y_2^{(2)} = \left[C_2 t + \frac{1}{7}(11C_2 + C_3)\right] e^t, \quad y_3^{(2)} = (7C_2 t + C_3)e^t.$$

Therefore, the general solution is

$$\left.\begin{aligned} y_1 &= y_1^{(1)} + y_1^{(2)} = \left[5C_2 t + \frac{1}{7}(13C_2 + 5C_3)\right] e^t \\ y_2 &= y_2^{(1)} + y_2^{(2)} = C_1 e^{6t} + \left[C_2 t + \frac{1}{7}(11C_2 + C_3)\right] e^t \\ y_3 &= y_3^{(1)} + y_3^{(2)} = 2C_1 e^{6t} + (7C_2 t + C_3)e^t \end{aligned}\right\}$$

Case III. Assume that r_1 is a real root of the characteristic equation (21) and the remaining two roots r_2, r_3 are complex conjugate.

$$r_2 = \alpha + i\beta, \qquad r_3 = \alpha - i\beta$$

Replacing in (20) r by r_1 and solving system (20), we obtain numbers λ_1, μ_1, ν_1. Thus, we have a system of particular solutions given by

$$y_1^{(1)} = C_1\lambda_1 e^{r_1 t} \qquad y_2^{(1)} = C_1\mu_1 e^{r_1 t} \qquad y_3^{(1)} = C_1\nu_1 e^{r_1 t}$$

Now, replacing in (20) r by $r_2 = \alpha + i\beta$ and solving (20), we get λ_2, μ_2, ν_2. Therefore, we have a system of particular solutions given by

$$y_1^{(2)} = \lambda_2 e^{(\alpha+i\beta)t} \qquad y_2^{(2)} = \mu_2 e^{(\alpha+i\beta)t} \qquad y_3^{(2)} = \nu_2 e^{(\alpha+i\beta)t}$$

which is equivalent to

$$y_1^{(2)} = \lambda_2 e^{\alpha t}(\cos\beta t + i\sin\beta t), \quad y_2^{(2)} = \mu_2 e^{\alpha t}(\cos\beta t + i\sin\beta t), \quad y_3^{(2)} = \nu_2 e^{\alpha t}(\cos\beta t + i\sin\beta t).$$

Lastly, replacing in (20) r by $r_3 = \alpha - i\beta$ and solving (20), we get λ_3, μ_3, ν_3. Therefore, we have a system of particular solutions given by

$$y_1^{(3)} = \lambda_3 e^{(\alpha - i\beta)t} \qquad y_2^{(3)} = \mu_3 e^{(\alpha - i\beta)t} \qquad y_3^{(3)} = \nu_3 e^{(\alpha - i\beta)t}$$

which is equivalent to

$$y_1^{(3)} = \lambda_3 e^{\alpha t}(\cos\beta t - i\sin\beta t), \quad y_2^{(3)} = \mu_3 e^{\alpha t}(\cos\beta t - i\sin\beta t), \quad y_3^{(3)} = \nu_3 e^{\alpha t}(\cos\beta t - i\sin\beta t).$$

If $z = a + bi$ is a complex number, then the real and imaginary parts of z denoted by Re z and Im z, respectively, are defined as follows:

$$a = \operatorname{Re} z, \qquad b = \operatorname{Im} z$$

Assume that $\overline{y_1^{(2)}}$, $\overline{y_2^{(2)}}$, $\overline{y_3^{(2)}}$ represent the real parts of $\left\{y_1^{(2)}, y_1^{(3)}\right\}, \left\{y_2^{(2)}, y_2^{(3)}\right\}, \left\{y_3^{(2)}, y_3^{(3)}\right\}$, respectively. Also, assume that $\overline{y_1^{(3)}}$, $\overline{y_2^{(3)}}$, $\overline{y_3^{(3)}}$ represent the imaginary parts of $\left\{y_1^{(2)}, y_1^{(3)}\right\}, \left\{y_2^{(2)}, y_2^{(3)}\right\}, \left\{y_3^{(2)}, y_3^{(3)}\right\}$.

Then the general solution of the system is

$$\left.\begin{aligned} y_1 &= C_1\lambda_1 e^{r_1 t} + C_2\overline{y_1^{(2)}} + C_3\overline{y_1^{(3)}} \\ y_2 &= C_1\mu_1 e^{r_1 t} + C_2\overline{y_2^{(2)}} + C_3\overline{y_2^{(3)}} \\ y_3 &= C_1\nu_1 e^{r_1 t} + C_2\overline{y_3^{(2)}} + C_3\overline{y_3^{(3)}} \end{aligned}\right\} \qquad (34)$$

Example 4. Integrate the following system:

$$\begin{aligned} \frac{dy_1}{dt} &= 8y_2 \\ \frac{dy_2}{dt} &= -2y_3 \\ \frac{dy_3}{dt} &= 2y_1 + 8y_2 - 2y_3 \end{aligned}$$

Let us form the system of type (20)

$$\left.\begin{aligned} -r\lambda + 8\mu \qquad\qquad &= 0 \\ -r\mu - 2\nu &= 0 \\ 2\lambda + 8\mu + (-2 - r)\nu &= 0 \end{aligned}\right\} \qquad (35)$$

and write the characteristic equation

$$\Delta = \begin{vmatrix} -r & 8 & 0 \\ 0 & -r & -2 \\ 2 & 8 & -2-r \end{vmatrix} = -r\begin{vmatrix} -r & -2 \\ 8 & -2-r \end{vmatrix} - 8\begin{vmatrix} 0 & -2 \\ 2 & -2-r \end{vmatrix} = -r[-r(-2-r) + 16] - 8(4)$$
$$= -r(r^2 + 2r + 16) - 32 = -r^3 - 2r^2 - 16r - 32 = 0 \Rightarrow r^3 + 2r^2 + 16r + 32 = 0.$$

Thus, we get

$$(r + 2)(r^2 + 16) = 0$$

$$r_1 = -2, \qquad r_2 = 4i, \qquad r_3 = -4i$$

If $r = r_1$ = -2, then from (35), we get the system

$$\left.\begin{aligned} 2\lambda_1 + 8\mu_1 \qquad\quad &= 0 \\ 2\mu_1 - 2\nu_1 &= 0 \\ 2\lambda_1 + 8\mu_1 \qquad\quad &= 0 \end{aligned}\right\}$$

whose solution is

$$\begin{pmatrix} \lambda_1 \\ \mu_1 \\ \nu_1 \end{pmatrix} = A \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix}, \qquad A = \nu_1, \ A \in \mathbb{R}.$$

If A = 1, then

$$\lambda_1 = -4, \qquad \mu_1 = 1, \qquad \nu_1 = 1$$

$$y_1^{(1)} = -4e^{-2t}, \quad y_2^{(1)} = e^{-2t}, \quad y_3^{(1)} = e^{-2t}$$

If $r = r_2 = 4i$, then from (35), we get the system

$$\left.\begin{aligned} -4i\lambda_2 + 8\mu_2 \qquad\qquad\qquad &= 0 \\ -4i\mu_2 - 2\nu_2 \qquad\quad &= 0 \\ 2\lambda_2 + 8\mu_2 + (-2-4i)\,\nu_2 &= 0 \end{aligned}\right\}$$

whose solution is

$$\begin{pmatrix} \lambda_2 \\ \mu_2 \\ \nu_2 \end{pmatrix} = A_1 \begin{pmatrix} 1 \\ -1/2i \\ 1 \end{pmatrix}, \qquad A_1 = \nu_2, \ A_1 \in \mathbb{C}.$$

If $A_1 = 2i$, then

$$\lambda_2 = 2i, \qquad \mu_2 = -1, \qquad \nu_2 = 2i$$

$$y_1^{(2)} = 2ie^{4it}, \quad y_2^{(2)} = -e^{4it}, \quad y_3^{(2)} = 2ie^{4it}$$

$$y_1^{(2)} = 2i(\cos 4t + i\sin 4t), \quad y_2^{(2)} = -\cos 4t - i\sin 4t\,, \quad y_3^{(2)} = 2i(\cos 4t + i\sin 4t)$$

$$y_1^{(2)} = -2\sin 4t + i(2\cos 4t), \quad y_2^{(2)} = -\cos 4t - i\sin(4t)\,, \quad y_3^{(2)} = -2\sin 4t + i(2\cos 4t). \ (36)$$

Lastly, If $r = r_3 = -4i$, then from (35), we get the system

$$\left.\begin{aligned} 4i\lambda_3 + 8\mu_3 \qquad\qquad\qquad &= 0 \\ 4i\mu_3 - 2\nu_3 \qquad\quad &= 0 \\ 2\lambda_3 + 8\mu_3 + (-2+4i)\,\nu_3 &= 0 \end{aligned}\right\}$$

whose solution is

$$\begin{pmatrix} \lambda_3 \\ \mu_3 \\ \nu_3 \end{pmatrix} = A_2 \begin{pmatrix} 1 \\ 1/2i \\ 1 \end{pmatrix}, \qquad A_2 = \nu_3, \ A_2 \in \mathbb{C}.$$

If $A_2 = 2i$, then

$$\lambda_3 = 2i, \qquad \mu_3 = 1, \qquad \nu_3 = 2i$$

$$y_1^{(3)} = 2ie^{-4it}, \quad y_2^{(3)} = e^{-4it}, \quad y_3^{(3)} = 2ie^{-4it}$$

$$y_1^{(3)} = 2i(\cos 4t - i\sin 4t), \quad y_2^{(3)} = \cos 4t - i\sin 4t\,, \quad y_3^{(3)} = 2i(\cos 4t - i\sin 4t)$$

$$y_1^{(3)} = 2\sin 4t + i(2\cos 4t), \quad y_2^{(3)} = \cos 4t - i\sin(4t)\,, \quad y_3^{(3)} = 2\sin 4t + i(2\cos 4t). \ (37)$$

Note that the corresponding real parts of (36) and (37) have opposite signs. The solutions of the system will be equivalent once $\overline{y_1^{(2)}}, \ \overline{y_2^{(2)}}, \ \overline{y_3^{(2)}}$ are replaced in (34) by either the positive values or the negative ones. For this reason, we can say that the real solutions $\overline{y_1^{(2)}}, \ \overline{y_2^{(2)}}, \ \overline{y_3^{(2)}}$ of $\left\{ y_1^{(2)}, \ y_1^{(3)} \right\}, \left\{ y_2^{(2)}, \ y_2^{(3)} \right\}, \left\{ y_3^{(2)}, \ y_3^{(3)} \right\}$ are, respectively,

$$\overline{y_1^{(2)}} = 2\sin 4t\,, \quad \overline{y_2^{(2)}} = \cos 4t, \quad \overline{y_3^{(2)}} = 2\sin 4t\,.$$

The imaginary solutions $\overline{y_1^{(3)}}, \ \overline{y_2^{(3)}}, \ \overline{y_3^{(3)}}$ of $\left\{ y_1^{(2)}, \ y_1^{(3)} \right\}, \left\{ y_2^{(2)}, \ y_2^{(3)} \right\}, \left\{ y_3^{(2)}, \ y_3^{(3)} \right\}$ are given by

$$\overline{y_1^{(3)}} = 2\cos 4t\,, \quad \overline{y_2^{(3)}} = -\sin 4t\,, \quad \overline{y_3^{(3)}} = 2\cos 4t\,.$$

Therefore, the general solution of the given system is

$$\left.\begin{aligned} y_1 &= C_1 y_1^{(1)} + C_2 \overline{y_1^{(2)}} + C_3 \overline{y_1^{(3)}} = -4C_1 e^{-2t} + 2C_2 \sin 4t + 2C_3 \cos 4t \\ y_2 &= C_1 y_2^{(1)} + C_2 \overline{y_2^{(2)}} + C_3 \overline{y_2^{(3)}} = C_1 e^{-2t} + C_2 \cos 4t - C_3 \sin 4t \\ y_3 &= C_1 y_3^{(1)} + C_2 \overline{y_3^{(2)}} + C_3 \overline{y_3^{(3)}} = C_1 e^{-2t} + 2C_2 \sin 4t + 2C_3 \cos 4t \end{aligned}\right\}$$

Method for Integration of a System of Three Linear Nonhomogeneous DE with Constant Coefficients (Method of Variation of Parameters)

Assume that for the given system

$$\left.\begin{aligned} &\frac{dy_1}{dt} + a_1 y_1 + b_1 y_2 + e_1 y_3 = f_1(t) \qquad (38A) \\ &\frac{dy_2}{dt} + a_2 y_1 + b_2 y_2 + e_2 y_3 = f_2(t) \qquad (38B) \\ &\frac{dy_3}{dt} + a_3 y_1 + b_3 y_2 + e_3 y_3 = f_3(t) \qquad (38C) \end{aligned}\right\} \qquad (38)$$

a general solution of the corresponding homogeneous system has been found.

By using the method of variation of parameters, let us assume that C_1, C_2, C_3 are functions of t.

$$\left.\begin{aligned} y_1 &= C_1(t)y_1^{(1)} + C_2(t)y_1^{(2)} + C_3(t)y_1^{(3)} \\ y_2 &= C_1(t)y_2^{(1)} + C_2(t)y_2^{(2)} + C_3(t)y_2^{(3)} \\ y_3 &= C_1(t)y_3^{(1)} + C_2(t)y_3^{(2)} + C_3(t)y_3^{(3)} \end{aligned}\right\} \qquad (39)$$

In order to compute the values of $C_1(t)$, $C_2(t)$, $C_3(t)$, find the derivative of y_1 with respect to t

$$y_1' = C_1 y_1^{(1)'} + C_1' y_1^{(1)} + C_2 y_1^{(2)'} + C_2' y_1^{(2)} + C_3 y_1^{(3)'} + C_3' y_1^{(3)} \qquad (40)$$

By substituting (39) and (40) into (38A), we have

$$\begin{aligned} C_1' y_1^{(1)} + C_2' y_1^{(2)} + C_3' y_1^{(3)} + C_1\left[y_1^{(1)'} + a_1 y_1^{(1)} + b_1 y_2^{(1)} + e_1 y_3^{(1)}\right] \\ + C_2\left[y_1^{(2)'} + a_1 y_1^{(2)} + b_1 y_2^{(2)} + e_1 y_3^{(2)}\right] + C_3\left[y_1^{(3)'} + a_1 y_1^{(3)} + b_1 y_2^{(3)} + e_1 y_3^{(3)}\right] \\ = f_1(t). \end{aligned}$$

All sums in brackets are zero (they represent the expressions of the corresponding homogeneous system). Thus, we obtain

$$C_1' y_1^{(1)} + C_2' y_1^{(2)} + C_3' y_1^{(3)} = f_1(t)$$

In a similar way, from (38B) and (38C), we get the remaining two equations with which we can then form the system

$$\left.\begin{aligned} C_1' y_1^{(1)} + C_2' y_1^{(2)} + C_3' y_1^{(3)} &= f_1(t) \\ C_1' y_2^{(1)} + C_2' y_2^{(2)} + C_3' y_2^{(3)} &= f_2(t) \\ C_1' y_3^{(1)} + C_2' y_3^{(2)} + C_3' y_3^{(3)} &= f_3(t) \end{aligned}\right\} \qquad (41)$$

System (41) with unknowns C_1', C_2', C_3' has a solution since its determinant

$$\mathrm{Det(A)} = \begin{vmatrix} y_1^{(1)} & y_1^{(2)} & y_1^{(3)} \\ y_2^{(1)} & y_2^{(2)} & y_2^{(3)} \\ y_3^{(1)} & y_3^{(2)} & y_3^{(3)} \end{vmatrix} \neq 0.$$

This is due to the fact that the particular solutions of the corresponding homogeneous system are linearly independent (see [1].)

Once C_1', C_2', C_3' are found, by using integration, we can determine C_1, C_2, C_3 which permits us to write the general solution (39) of (38).

Example 5. Integrate the system

$$\begin{aligned}\frac{dy_1}{dt} &= 2y_1 + y_2 - 2y_3 + (2-t) \\ \frac{dy_2}{dt} &= -y_1 \qquad\qquad + 1 \\ \frac{dy_3}{dt} &= y_1 + y_2 - y_3 + (1-t)\end{aligned} \tag{42}$$

Firstly, let us solve the homogeneous system

$$\begin{aligned}\frac{dy_1}{dt} &= 2y_1 + y_2 - 2y_3 \\ \frac{dy_2}{dt} &= -y_1 \\ \frac{dy_3}{dt} &= y_1 + y_2 - y_3\end{aligned}$$

Let us form the system of type (20)

$$\left.\begin{aligned}(2-r)\lambda + \mu \qquad - 2\nu &= 0 \\ -\lambda - r\mu \qquad\qquad &= 0 \\ \lambda + \mu + (-1-r)\nu &= 0\end{aligned}\right\} \tag{43}$$

The characteristic equation is

$$\begin{aligned}\Delta = \begin{vmatrix} 2-r & 1 & -2 \\ -1 & -r & 0 \\ 1 & 1 & -1-r \end{vmatrix} &= \begin{vmatrix} 1 & -2 \\ 1 & -1-r \end{vmatrix} - r\begin{vmatrix} 2-r & -2 \\ 1 & -1-r \end{vmatrix} \\ &= (-1-r+2) - r[(2-r)(-1-r)+2] = (1-r) - r(-r+r^2) \\ &= -r^3 + r^2 - r + 1 = 0 \ \Rightarrow r^3 - r^2 + r - 1 = 0.\end{aligned}$$

So, we obtain

$$(r-1)(r^2+1) = 0$$

$$r_1 = 1, \qquad r_2 = i, \qquad r_3 = -i$$

If $r = r_1 = 1$, then from (43), we get the system

$$\left.\begin{aligned}\lambda_1 + \mu_1 - 2\nu_1 &= 0 \\ -\lambda_1 - \mu_1 \qquad &= 0 \\ \lambda_1 + \mu_1 - 2\nu_1 &= 0\end{aligned}\right\}$$

whose solution is

$$\begin{pmatrix}\lambda_1 \\ \mu_1 \\ \nu_1\end{pmatrix} = A\begin{pmatrix}-1 \\ 1 \\ 0\end{pmatrix}, \qquad A = \mu_1, \ A \in \mathbb{R}.$$

If $A = 1$, then

$$\lambda_1 = -1, \qquad \mu_1 = 1, \qquad \nu_1 = 0$$

$$y_1^{(1)} = -e^t, \quad y_2^{(1)} = e^t, \quad y_3^{(1)} = 0$$

If $r = r_2 = i$, then from (43), we get the system

$$\left.\begin{array}{r} (2-i)\lambda_2 + \mu_2 \qquad - 2\nu_2 = 0 \\ -\lambda_2 - i\mu_2 \qquad\qquad = 0 \\ \lambda_2 + \mu_2 + (-1-i)\,\nu_2 = 0 \end{array}\right\}$$

whose solution is

$$\begin{pmatrix} \lambda_2 \\ \mu_2 \\ \nu_2 \end{pmatrix} = A_1 \begin{pmatrix} 1 \\ -1/i \\ 1 \end{pmatrix}, \qquad A_1 = \nu_2, \;\; A_1 \in \mathbb{C}.$$

If $A_1 = i$, then

$$\lambda_2 = i, \qquad \mu_2 = -1, \qquad \nu_2 = i$$

$$y_1^{(2)} = ie^{it}, \quad y_2^{(2)} = -e^{it}, \quad y_3^{(2)} = ie^{it}$$

$$y_1^{(2)} = i(\cos t + i\sin t), \quad y_2^{(2)} = -\cos t - i\sin t\,, \quad y_3^{(2)} = i(\cos t + i\sin t)$$

$$y_1^{(2)} = -\sin t + i\cos t\,, \quad y_2^{(2)} = -\cos t - i\sin t\,, \quad y_3^{(2)} = -\sin t + i\cos t.$$

Lastly, If $r = r_3 = -i$, then from (43), we get the system

$$\left.\begin{array}{r} (2+i)\lambda_3 + \mu_3 \qquad - 2\nu_3 = 0 \\ -\lambda_3 + i\mu_3 \qquad\qquad = 0 \\ \lambda_3 + \mu_3 + (-1+i)\,\nu_3 = 0 \end{array}\right\}$$

whose solution is

$$\begin{pmatrix} \lambda_3 \\ \mu_3 \\ \nu_3 \end{pmatrix} = A_2 \begin{pmatrix} 1 \\ 1/i \\ 1 \end{pmatrix}, \qquad A_2 = \nu_3, \;\; A_2 \in \mathbb{C}.$$

If $A_2 = i$, then

$$\lambda_3 = i, \qquad \mu_3 = 1, \qquad \nu_3 = i$$

$$y_1^{(3)} = ie^{-it}, \quad y_2^{(3)} = e^{-it}, \quad y_3^{(3)} = ie^{-it}$$

$$y_1^{(3)} = i(\cos t - i\sin t), \quad y_2^{(3)} = \cos t - i\sin t\,, \quad y_3^{(3)} = i(\cos t - i\sin t)$$

$$y_1^{(3)} = \sin t + i\cos t\,, \quad y_2^{(3)} = \cos t - i\sin t\,, \quad y_3^{(3)} = \sin t + i\cos t.$$

Therefore,

$$\overline{y_1^{(2)}} = \sin t\,, \quad \overline{y_2^{(2)}} = \cos t, \quad \overline{y_3^{(2)}} = \sin t\,,$$

$$\overline{y_1^{(3)}} = \cos t, \quad \overline{y_2^{(3)}} = -\sin t, \quad \overline{y_3^{(3)}} = \cos t.$$

Therefore, from the general solution of the corresponding homogeneous system, we get

$$\left.\begin{aligned} y_1 &= C_1 y_1^{(1)} + C_2\overline{y_1^{(2)}} + C_3\overline{y_1^{(3)}} = -C_1(t)e^t + C_2(t)\sin t + C_3(t)\cos t \\ y_2 &= C_1 y_2^{(1)} + C_2\overline{y_2^{(2)}} + C_3\overline{y_2^{(3)}} = C_1(t)e^t + C_2(t)\cos t - C_3(t)\sin t \\ y_3 &= C_1 y_3^{(1)} + C_2\overline{y_3^{(2)}} + C_3\overline{y_3^{(3)}} = C_2(t)\sin t + C_3(t)\cos t \end{aligned}\right\} \qquad (44)$$

Now, let us form the system

$$\left.\begin{aligned} -e^t C_1' + \sin t\, C_2' + \cos t\, C_3' &= 2 - t \qquad (45A) \\ e^t C_1' + \cos t\, C_2' - \sin t\, C_3' &= 1 \qquad (45B) \\ \sin t\, C_2' + \cos t\, C_3' &= 1 - t \qquad (45C) \end{aligned}\right\} \qquad (45)$$

By subtracting (45*C*) from (45*A*), we have

$$C_1' = -e^{-t} \qquad (46)$$

$$C_1(t) = e^{-t} + C_1. \qquad (46A)$$

According to (45*C*)

$$C_3' = \frac{1 - t - \sin t\, C_2'}{\cos t} \qquad (45D)$$

Substituting (46) and (45*D*) into (45B), we get

$$-1 + \cos t\, C_2' - \frac{\sin t\,(1 - t - \sin t\, C_2')}{\cos t} = 1$$

$$C_2' = 2\cos t + \sin t - t\sin t \qquad (47)$$

$$C_2(t) = 2\sin t - \cos t - \int t\sin t\, dt$$

$$C_2(t) = 2\sin t - \cos t + t\cos t - \sin t + C_2$$

$$C_2(t) = \sin t - \cos t + t\cos t + C_2 \qquad (47A)$$

By substituting (47) into (45*D*), we have

$$C_3' = \frac{1 - t - \sin t\,(2\cos t + \sin t - t\sin t)}{\cos t}$$

$$C_3' = \cos t - 2\sin t - t\cos t$$

$$C_3(t) = \sin t + 2\cos t - \int t\cos t\, dt$$

$$C_3(t) = \sin t + 2\cos t - t\sin t - \cos t + C_3$$

$$C_3(t) = \sin t + \cos t - t\sin t + C_3 \qquad (48)$$

Substituting (46*A*), (47*A*), (48) into (44), we obtain

$$\left.\begin{aligned} y_1 &= (-e^{-t} - C_1)e^t + (\sin t - \cos t + t\cos t + C_2)\sin t + (\sin t + \cos t - t\sin t + C_3)\cos t \\ y_2 &= (e^{-t} + C_1)e^t + (\sin t - \cos t + t\cos t + C_2)\cos t - (\sin t + \cos t - t\sin t + C_3)\sin t \\ y_3 &= (\sin t - \cos t + t\cos t + C_2)\sin t + (\sin t + \cos t - t\sin t + C_3)\cos t \end{aligned}\right\}$$

Thus, from the last system, we get the general solution

$$\left.\begin{aligned} y_1 &= -C_1 e^t + C_2\sin t + C_3\cos t \\ y_2 &= C_1 e^t + C_2\cos t - C_3\sin t + t \\ y_3 &= C_2\sin t + C_3\cos t + 1 \end{aligned}\right\}$$

Solving a System of Two Linear Nonhomogeneous DE with Given Initial Conditions by Using the Laplace Transform

Assume we need to find the solution of the system

$$\left.\begin{aligned} \frac{dy_1}{dt} &= a_1 y_1 + b_1 y_2 + f_1(t) \\ \frac{dy_2}{dt} &= a_2 y_1 + b_2 y_2 + f_2(t) \end{aligned}\right\} \qquad (49)$$

with initial conditions

$$y_1(0) = y_1^{(0)}, \quad y_2(0) = y_2^{(0)}. \qquad (50)$$

Assume now that $Y_1(s), Y_2(s)$ are the corresponding $\mathcal{L}$-transforms of $y_1(t), y_2(t)$. Under these conditions, we have

$$\mathcal{L}\{y_1(t)\} = Y_1(s), \qquad \mathcal{L}\{y_2(t)\} = Y_2(s)$$

$$\mathcal{L}\{y_1'(t)\} = s\mathcal{L}\{y_1(t)\} - y_1(0), \qquad \mathcal{L}\{y_2'(t)\} = s\mathcal{L}\{y_2(t)\} - y_2(0)$$

$$\mathcal{L}\{f_1(t)\} = F_1(s), \qquad \mathcal{L}\{f_2(t)\} = F_2(s)$$

Thus, by applying the Laplace transform to both sides of each equation of (49), we have the operational system

$$\left.\begin{aligned} \mathcal{L}\{y_1'(t)\} &= a_1\mathcal{L}\{y_1(t)\} + b_1\mathcal{L}\{y_2(t)\} + \mathcal{L}\{f_1(t)\} \\ \mathcal{L}\{y_2'(t)\} &= a_2\mathcal{L}\{y_1(t)\} + b_2\mathcal{L}\{y_2(t)\} + \mathcal{L}\{f_2(t)\} \end{aligned}\right\}$$

which is equivalent to

$$\left.\begin{aligned} sY_1(s) &= a_1 Y_1(s) + b_1 Y_2(s) + F_1(s) + y_1(0) \\ sY_2(s) &= a_2 Y_1(s) + b_2 Y_2(s) + F_2(s) + y_2(0) \end{aligned}\right\}. \qquad (51)$$

(51) is an algebraic system of two linear equations with unknowns $Y_1(s)$, $Y_2(s)$. Once (51) is solved, the inverse Laplace transform $\mathcal{L}^{-1}$ is applied to the values of $Y_1(s)$, $Y_2(s)$ in order to obtain the solution $y_1(t)$, $y_2(t)$ of system (49) with initial conditions (50).

Example 6. Solve the system of DE with the given initial conditions

$$\left.\begin{aligned} \frac{dy_1}{dt} + y_2 = 0 \\ \frac{dy_2}{dt} + y_1 = 0 \end{aligned}\right\}, \qquad y_1(0) = 2, \quad y_2(0) = 0.$$

Firstly, let us write the system in the form

$$\left.\begin{aligned} \frac{dy_1}{dt} &= \qquad -y_2 \\ \frac{dy_2}{dt} &= -y_1 \end{aligned}\right\}$$

It follows then that

$$\left.\begin{aligned} \mathcal{L}\{y_1'(t)\} &= \qquad\qquad -\mathcal{L}\{y_2(t)\} \\ \mathcal{L}\{y_2'(t)\} &= -\mathcal{L}\{y_1(t)\} \end{aligned}\right\}$$

which can be written in the form

$$\left.\begin{aligned} sY_1(s) - 2 &= \qquad\quad -Y_2(s) \\ sY_2(s) - 0 &= -Y_1(s) \end{aligned}\right\}$$

or simply

$$\left.\begin{aligned} sY_1(s) + Y_2(s) &= 2 \\ Y_1(s) + sY_2(s) &= 0 \end{aligned}\right\}.$$

Applying Cramer’s rule, this system has a solution given by

$$Y_1(s) = \frac{2s}{s^2 - 1}, \qquad Y_2(s) = -\frac{2}{s^2 - 1}.$$

Therefore, according to Table 1,

$$y_1(t) = \mathcal{L}^{-1}\{\mathcal{L}\{y_1(t)\}\} = \mathcal{L}^{-1}\{Y_1(s)\} = \mathcal{L}^{-1}\left\{\frac{2s}{s^2 - 1}\right\} = 2\mathcal{L}^{-1}\left\{\frac{s}{s^2 - 1}\right\} = 2\cosh t,$$

$$y_2(t) = \mathcal{L}^{-1}\{\mathcal{L}\{y_2(t)\}\} = \mathcal{L}^{-1}\{Y_2(s)\} = \mathcal{L}^{-1}\left\{-\frac{2}{s^2 - 1}\right\} = -2\mathcal{L}^{-1}\left\{\frac{1}{s^2 - 1}\right\} = -2\sinh t$$

Example 7. Solve the system of DE with the given initial conditions

$$\left.\begin{aligned} \frac{dy_1}{dt} + 2y_2 &= 3t \\ \frac{dy_2}{dt} - 2y_1 &= 4 \end{aligned}\right\}, \qquad y_1(0) = 2, \quad y_2(0) = 3.$$

Let us write the system in the form

$$\left.\begin{aligned} \frac{dy_1}{dt} &= \qquad -2y_2 + 3t \\ \frac{dy_2}{dt} &= 2y_1 \qquad\quad + 4 \end{aligned}\right\}$$

It follows that

$$\left.\begin{aligned} \mathcal{L}\{y_1'(t)\} &= \qquad\qquad -2\mathcal{L}\{y_2(t)\} + 3\mathcal{L}\{t\} \\ \mathcal{L}\{y_2'(t)\} &= 2\mathcal{L}\{y_1(t)\} \qquad\qquad + 4\mathcal{L}\{1\} \end{aligned}\right\}$$

which is equivalent to

$$\left.\begin{aligned} sY_1(s) - 2 &= \qquad\quad -2Y_2(s) + \frac{3}{s^2} \\ sY_2(s) - 3 &= 2Y_1(s) \qquad\quad + \frac{4}{s} \end{aligned}\right\}$$

or simply

$$\left.\begin{aligned} sY_1(s) + 2Y_2(s) &= \frac{2s^2 + 3}{s^2} \\ -2Y_1(s) + sY_2(s) &= \frac{3s + 4}{s} \end{aligned}\right\}$$

Applying Cramer's rule, this system has a solution given by

$$Y_1(s) = \frac{2s^2 - 6s - 5}{s(s^2 + 4)}, \quad Y_2(s) = \frac{3s^3 + 8s^2 + 6}{s^2(s^2 + 4)}. \qquad (52)$$

Now, we need to transform the last two solutions as follows:

$$\frac{2s^2 - 6s - 5}{s(s^2 + 4)} = \frac{2s}{s^2 + 4} - \frac{6}{s^2 + 4} - \frac{5}{s(s^2 + 4)}.$$

We need to use the partial fraction expansion of a rational expression

$$\frac{5}{s(s^2 + 4)} = \frac{A}{s} + \frac{Bs + C}{(s^2 + 4)}$$

whose values for *A*, *B*, *C* are

$$A = \frac{5}{4}, \quad B = -\frac{5}{4}, \quad C = 0$$

which gives

$$\frac{5}{s(s^2+4)} = \frac{5}{4s} - \frac{5s}{4(s^2+4)}.$$

On the other hand,

$$\frac{3s^3+8s^2+6}{s^2(s^2+4)} = \frac{3s}{s^2+4} + \frac{8}{s^2+4} + \frac{6}{s^2(s^2+4)}$$

We use again the partial fraction expansion of a rational expression to obtain

$$\frac{6}{s^2(s^2+4)} = \frac{3}{2}\left(\frac{1}{s^2} - \frac{1}{s^2+4}\right)$$

Now, we can rewrite (52) as follows:

$$Y_1(s) = \frac{2s}{s^2+4} - \frac{6}{s^2+4} - \frac{5}{4s} + \frac{5s}{4(s^2+4)} = \frac{13s}{4(s^2+4)} - \frac{6}{s^2+4} - \frac{5}{4s},$$

$$Y_2(s) = \frac{3s}{s^2+4} + \frac{8}{s^2+4} + \frac{3}{2s^2} - \frac{3}{2(s^2+4)} = \frac{3s}{s^2+4} + \frac{13}{2(s^2+4)} + \frac{3}{2s^2}.$$

Therefore, according to table 1,

$$y_1(t) = \mathcal{L}^{-1}\{\mathcal{L}\{y_1(t)\}\} = \mathcal{L}^{-1}\{Y_1(s)\} = \frac{13}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} - 3\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\} - \frac{5}{4}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$$
$$= \frac{13}{4}\cos 2t - 3\sin 2t - \frac{5}{4},$$

$$y_2(t) = \mathcal{L}^{-1}\{\mathcal{L}\{y_2(t)\}\} = \mathcal{L}^{-1}\{Y_2(s)\} = 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} + \frac{13}{4}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\} + \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$
$$= 3\cos 2t + \frac{13}{4}\sin 2t + \frac{3}{2}t\,.$$

Exercises.

1. Solve the initial value problems

a)

$$y'' - y' = 2\sin t, \qquad y(0) = 2, \quad y'(0) = 0$$

Solution.

$$y = \cos t - \sin t + e^t$$

b)

$$y''' + y'' - 2y = 5e^t, \qquad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 2$$

Solution.

$$y = te^t$$

2. Integrate the following system by reducing it to a second order DE:

$$\frac{dy_1}{dt} = 3y_1 + 2y_2 + e^t$$
$$\frac{dy_2}{dt} = y_1 + 2y_2 + e^{4t}$$

Solution.

$$y_1 = C_1e^t + C_2e^{4t} + \frac{2}{3}te^{4t} + \frac{1}{3}te^t$$
$$y_2 = -C_1e^t + \frac{1}{2}C_2e^{4t} + \frac{1}{3}te^{4t} + \frac{1}{3}e^{4t} - \frac{1}{3}te^t - \frac{1}{3}e^t$$

3. Integrate the following homogeneous systems:

a)

$$\frac{dy_1}{dt} = y_2 + y_3$$
$$\frac{dy_2}{dt} = 3y_1 + y_3$$
$$\frac{dy_3}{dt} = 3y_1 + y_2$$

Solution.

$$\left.\begin{aligned} y_1 &= -C_2e^{-2t} + \frac{2}{3}C_3e^{3t} \\ y_2 &= -C_1e^{-t} + C_2e^{-2t} + C_3e^{3t} \\ y_3 &= C_1e^{-t} + C_2e^{-2t} + C_3e^{3t} \end{aligned}\right\}$$

b)

$$\frac{dy_1}{dt} = y_2 + y_3$$
$$\frac{dy_2}{dt} = y_1 + y_3$$
$$\frac{dy_3}{dt} = y_1 + y_2$$

Solution.

$$\left.\begin{aligned} y_1 &= C_1e^{2t} - (C_2 + C_3)e^{-t} \\ y_2 &= C_1e^{2t} + C_2e^{-t} \\ y_3 &= C_1e^{2t} + C_3e^{-t} \end{aligned}\right\}$$

c)

$$\begin{aligned}\frac{dy_1}{dt} &= 2y_1 + y_2\\ \frac{dy_2}{dt} &= y_1 + 3y_2 - y_3\\ \frac{dy_3}{dt} &= -y_1 + 2y_2 + 3y_3\end{aligned}$$

Solution.

$$\left.\begin{aligned} y_1 &= C_1e^{2t} + C_2e^{3t}\cos t + C_3e^{3t}\sin t\\ y_2 &= C_2e^{3t}(\cos t - \sin t) + C_3e^{3t}(\sin t + \cos t)\\ y_3 &= C_1e^{2t} + C_2e^{3t}(2\cos t + \sin t) + C_3e^{3t}(2\sin t - \cos t)\end{aligned}\right\}$$

4. Integrate the following nonhomogeneous system:

$$\begin{aligned}\frac{dy_1}{dt} &= -y_1 + y_2 + y_3 + e^t\\ \frac{dy_2}{dt} &= y_1 - y_2 + y_3 + e^{3t}\\ \frac{dy_3}{dt} &= y_1 + y_2 + y_3 + 4\end{aligned}$$

Solution.

$$\left.\begin{aligned} y_1 &= \frac{1}{3}C_1e^{-t} + \frac{1}{6}C_2e^{2t} + \frac{1}{2}C_3e^{-2t} + \frac{1}{6}e^t + \frac{3}{20}e^{3t} - 2\\ y_2 &= \frac{1}{3}C_1e^{-t} + \frac{1}{6}C_2e^{2t} - \frac{1}{2}C_3e^{-2t} - \frac{1}{6}e^t + \frac{7}{20}e^{3t} - 2\\ y_3 &= -\frac{1}{3}C_1e^{-t} + \frac{1}{3}C_2e^{2t} - \frac{1}{2}e^t + \frac{1}{4}e^{3t}\end{aligned}\right\}$$

5. Integrate the following systems with the given initial conditions:

a)

$$\left.\begin{aligned}\frac{dy_1}{dt} + y_1 - 2y_2 &= 0\\ \frac{dy_2}{dt} + y_1 + 4y_2 &= 0\end{aligned}\right\}, \qquad y_1(0) = y_2(0) = 1.$$

Solution.

$$\left.\begin{aligned} y_1(t) &= 4e^{-2t} - 3e^{-3t}\\ y_2(t) &= 3e^{-3t} - 2e^{-2t}\end{aligned}\right\}$$

b)

$$\left.\begin{array}{l}\dfrac{dy_1}{dt}+y_1=y_2+\ e^t\\\dfrac{dy_2}{dt}+y_2=y_1+\ e^t\end{array}\right\},\qquad y_1(0)=y_2(0)=1.$$

Solution.

$$\left.\begin{array}{l}y_1(t)=e^t\\y_2(t)=e^t\end{array}\right\}$$

References

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