Math 332: Ordinary Differential Equations
Lecture Notes

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Spring 2022

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Week 1, Monday: Separable equations

Homepage. Please see our course homepage. In particular, please follow the "Course information" link and carefully read the material there.

Text. Our text is *Differential equations and dynamical systems*, edition 3, by Lawrence Perko.

Overview. This is a course in differential equations for advanced undergraduates. Here is a rough outline of what we'll cover:

1. **Elementary methods.** We will spend two weeks to cover the major part of what one might do in a first course in differential equations in a lower-level course.
2. **Linear theory.** We will then learn how to solve systems of linear differential equations with constant coefficients. The key is exponentiation of matrices and the Jordan form.
3. **Local nonlinear theory.** Most nonlinear systems cannot be solved. So we are interested in describing the qualitative behavior of such systems. The main idea is to approximate nonlinear systems with linear systems. We will cover the important existence-uniqueness theorems.
4. **Global theory.** At the end of the course, we will study limits of trajectories and topological properties for systems of ODEs.

First goal: elementary methods. For this part of the course, we will not have a text. The main source for information will be the lecture notes and handouts. There is also plenty of material readily available online. My hope is that this will be a time to have fun doing lots of computations by hand. We will cover the six methods from the handout First recipes, starting with separable equations. In the following, we will generally think of y as a real-valued function of t.

I. Separable equations. A separable differential equation has the form (or can be manipulated to have the form)

$$p(y)\frac{dy}{dt} = q(t).$$

It is solved by integration:

$$\int p(y)\, dy = \int q(t)\, dt.$$

EXAMPLES. Consider the differential equation

$$y' = \frac{3t}{y}.$$

It's separable since we can get the ys on one side of the equality and the ts on the other:

$$yy' = 3t.$$

Integrate:

$$\int y(t)y'(t)\, dt = \int 3t\, dt.$$

Forgetting about constant until the end, the right-hand side is

$$\int 3t\, dt = \frac{3}{2}t^2$$

For the left-hand side, make the substitution $u = y(t)$. So $du = y'(t)\, dt$. Substituting gives:

$$\int y(t)y'(t)\, dt = \int u\, du = \frac{1}{2}u^2 = \frac{1}{2}y^2.$$

Setting the two sides equal and adding a constant gives the most general solution:

$$\frac{1}{2}y^2 = \frac{3}{2}t^2 + \tilde{c}$$

or, equivalently,

$$\boxed{y(t)^2 = 3t^2 + c}$$

for some constant c.

(An alternative way to integrate:

$$\int y\, dy = \int 3t\, dt \quad \Rightarrow \quad \frac{1}{2}y^2 = \frac{3}{2}t^2 + c.)$$

To find a particular solution, we can impose an initial condition. For instance, if $y(0) = 5$, then

$$25 = y(0)^2 = 3 \cdot 0^2 + c \quad \Rightarrow \quad c = 25,$$

and the solution is defined implicitly by

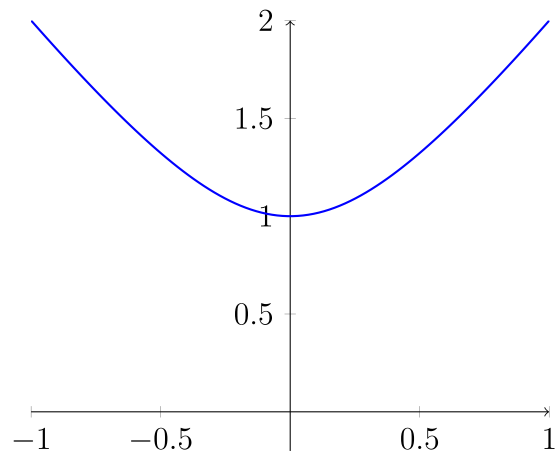
$$y(t)^2 = 3t^2 + 25.$$

Thus, $y(t) = \pm\sqrt{3t^2 + 25}$. Since we want $y(0) = 5$, we must choose the positive solution:

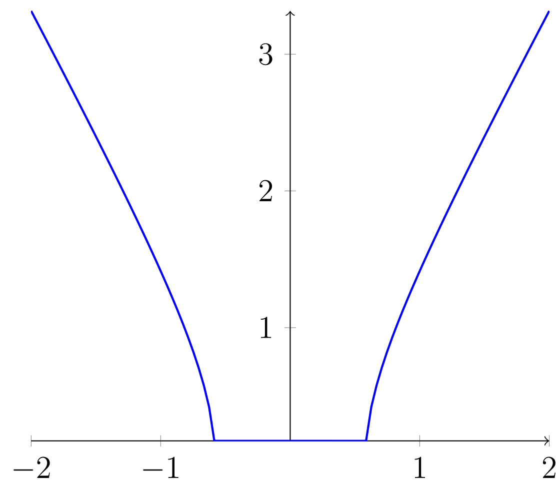
$$y(t) = \sqrt{3t^2 + 25}.$$

It is a solution for all $t \in \mathbb{R}$. If you initial condition were $y(0) = -5$, the solution would be $y(t) = -\sqrt{3t^2 + 25}$, again for all $t \in \mathbb{R}$.

There are, qualitatively, two types of behavior for solutions of this differential equation depending on whether c is positive or negative.



Graph of $y(t) = \sqrt{3t^2 + 1}$.



Graph of $y(t) = \sqrt{3t^2 - 1}$.

For an example where $c < 0$, suppose the initial condition is $y(-1) = \sqrt{2}$. Then

$$2 = y(-1)^2 = 3 \cdot 1^2 + c \quad \Rightarrow \quad c = -1,$$

and the implicit solution is

$$y(t)^2 = 3t^2 - 1.$$

The solution (with the given initial condition) is

$$y(t) = \sqrt{3t^2 - 1},$$

which makes sense for $3t^2 \geq 1$, i.e., $t \geq \sqrt{3}/3$ and $t \leq -\sqrt{3}/3$. Since our initial condition is at $t = -1$, the maximal interval for the solution is $(-\infty, -\sqrt{3}/3)$.

Exponential growth and decay model. Let $y(t)$ now denote the size of a population, varying over time. What happens if we assume that the rate of growth of the population is proportional to the size of the population? The rate of growth of the population is $y'(t)$ and the size of the population is $y(t)$. To say they are proportional is to say there is a constant r such that

$$y'(t) = ry(t).$$

This is a separable equation, which is easy to solve:

$$y'(t) = ry(t) \quad \Rightarrow \quad \frac{y'(t)}{y(t)} = r \quad \Rightarrow \quad \int \frac{y'(t)}{y(t)}\,dt = \int r\,dt.$$

Integrate, then solve for y:

$$\ln|y(t)| = rt + c \quad \Rightarrow \quad |y(t)| = e^{rt+c} = e^c e^{rt} = ae^{rt},$$

where a a positive constant. So the solution is

$$y(t) = \begin{cases} ae^{rt} & \text{if } y > 0 \\ -ae^{rt} & \text{if } y < 0. \end{cases}$$

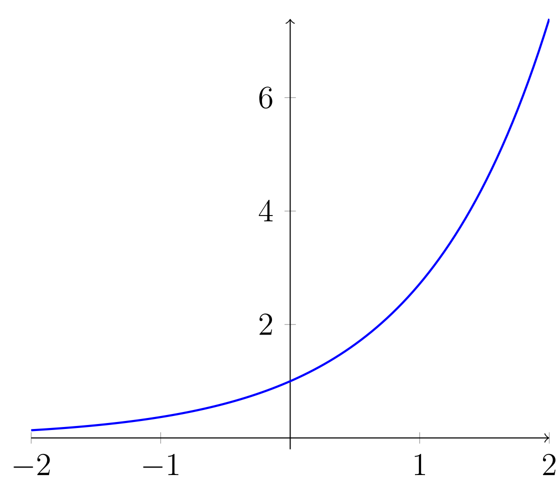
where a is positive. But we can combine these two solutions into the single solution $y(t) = ae^{rt}$ by letting a be any nonzero real number. Setting $t = 0$, we see

$$y(0) = ae^0 = a.$$

Hence, a is the initial population. So we might write the solution as

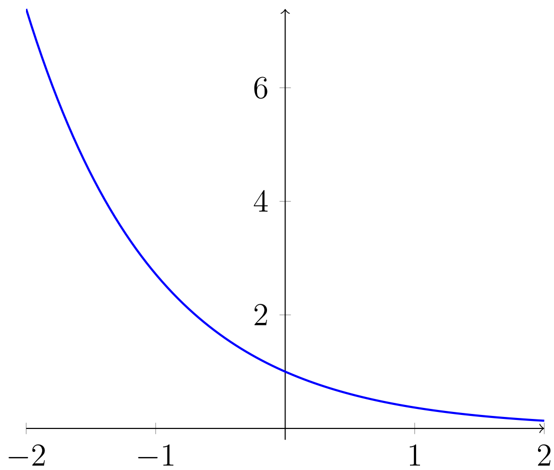
$$y(t) = y_0 e^{rt}.$$

For instance, if $y_0 = r = 1$, we get the picture below:



Graph of $y(t) = e^t$.

If $y_0 = 1$ and $r = -1$, we get:



Graph of $y(t) = e^t$.

In performing the integration, *we assumed that y was never zero in the range over which we integrated*. What if the initial condition is $y(t_0) = 0$ for some time t_0? One solution then is to take $y(t) = 0$ for all t. Again, the equation $y(t) = y_0 e^{rt}$ works. Is this the only solution? We'll focus on this question later in the course.

Example. If $y(t) = ae^{rt}$ with $y(0) = a \neq 0$ at what time t has the population doubled?

SOLUTION: The initial population size is a. So we are trying to find the time t when $y(t) = 2a$, so we need to solve

$$ae^{rt} = 2a.$$

Since $a \neq 0$, we need to solve

$$e^{rt} = 2$$

for t. Taking logs,

$$\ln(2) = \ln(e^{rt}) = rt.$$

Hence, assuming $r \neq 0$,

$$t = \frac{\ln(2)}{r}.$$

If $r = 0$, then $y(t) = a$ for all t, and the population never doubles.

Population model based on Newton's law of cooling. Suppose now that the rate of change of the population is governed by the differential equation

$$y'(t) = r(S - y(t))$$

where r and S are positive constants.

Problems:

1. When is the population increasing? Decreasing?

   ANSWER: We have

   $$y'(t) = r(S - y(t)) > 0 \quad \Leftrightarrow \quad S - y(t) > 0 \quad \Leftrightarrow \quad S > y(t).$$

   So the population is increasing whenever it's less that S and decreasing whenever it's larger than S.

2. What is the long-term behavior of the population?

   ANSWER: Given the answer to the previous problem it seems like the population should tend towards S.

3. Solve the equation assuming $y < S$.

   SOLUTION: The equation is separable:

   $$\begin{aligned} \int \frac{dy}{S-y} = \int r\,dt \quad &\Rightarrow \quad -\ln(S-y) = rt + c \\ &\Rightarrow \quad S - y = ae^{-rt} \\ &\Rightarrow \quad y = S - ae^{-rt}. \end{aligned}$$

   Note that $y(t) \to S$ as $t \to \infty$.

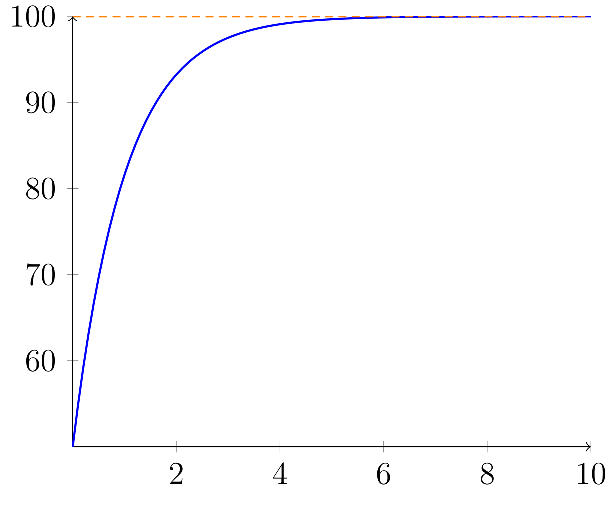
   Let's now make the initial population explicit in the solution. Say I is the initial population. Then

   $$I = y(0) = S - ae^0 = S - a \quad \Rightarrow \quad a = S - I.$$

   Our final form for the solution is

   $$\boxed{y(t) = S - (S - I)e^{-rt}}\,,$$

   where $I = y(0)$ is the initial population.



Graph of $y(t) = S - (S - I)e^{-rt}$ with $S = 100$, $I = 50$, and $r = 1$.

Week 1, Wednesday: Logistic equation. Homogeneity trick for separable equations

1. A. Separable equations.

Logistic growth model. Let $P(t)$ be the size of a population at time t, and let r and K be positive constants. The logistic growth model is the differential equation

$$P'(t) = rP(t)\left(1 - \frac{P(t)}{K}\right).$$

The constant r is the *growth rate* and K is the *carrying capacity* of the population. The differential equation says the growth in population is proportional to the size of the existing population with an extra factor to account for limited resources. When the population is small (when P is much smaller then K), we see $P' \approx rP$, which we've already seen leads to exponential growth. However, as P gets close to K over time, the factor $1 - P/K$ slows the growth.

Solution. The equation is separable and can be solved using integration using the technique of partial fractions.

$$P'(t) = rP(t)\left(1 - \frac{P(t)}{K}\right) \quad \Rightarrow \quad \frac{P'(t)}{P(t)\left(1 - \frac{P(t)}{K}\right)} = r.$$

The technique of partial fractions requires us to find constants A and B such that

$$\frac{1}{P(t)\left(1 - \frac{P(t)}{K}\right)} = \frac{A}{P(t)} + \frac{B}{1 - \frac{P(t)}{K}}. \tag{2.1}$$

We have

$$\frac{A}{P(t)} + \frac{B(t)}{1 - \frac{P(t)}{K}} = \frac{A\left(1 - \frac{P(t)}{K}\right) + BP(t)}{P(t)\left(1 - \frac{P(t)}{K}\right)}. \tag{2.2}$$

Comparing numerators in equations (2.1) and (2.2), we need to adjust A and B so that

$$1 = A\left(1 - \frac{P(t)}{K}\right) + BP(t).$$

Or, rearranging:

$$1 = A + \left(-\frac{A}{K} + B\right) P(t).$$

We get an equality if

$$A = 1 \quad \text{and} \quad -\frac{A}{K} + B = 0.$$

So $A = 1$ and $B = 1/K$. Therefore, we can write (double-check!):

$$\frac{1}{P(t)\left(1 - \frac{P(t)}{K}\right)} = \frac{1}{P(t)} + \frac{1/K}{1 - \frac{P(t)}{K}}. \tag{2.3}$$

Back to solving the differential equation:

$$\frac{P'(t)}{P(t)\left(1 - \frac{P(t)}{K}\right)} = r \quad \Rightarrow \quad \int \frac{dP}{P\left(1 - \frac{P}{K}\right)} = \int r\, dt$$

$$\Rightarrow \quad \int \frac{dP}{P(t)\left(1 - \frac{P}{K}\right)} = rt + \text{constant}.$$

For the left-hand side, use equation (2.3):

$$\begin{aligned}
\int \frac{dP}{P(t)\left(1 - \frac{P}{K}\right)}\, dt &= \int \left(\frac{1}{P(t)} + \frac{1/K}{1 - \frac{P(t)}{K}}\right) dP \\
&= \int \frac{dP}{P(t)} + \frac{1}{K} \int \frac{dP}{1 - \frac{P(t)}{K}} \\
&= \ln P(t) - \ln\left(1 - \frac{P(t)}{K}\right) + \text{constant}.
\end{aligned}$$

Here, we have assumed that $0 < P < K$ (how?). Exponentiate both sides to get

$$P(t)\left(1 - \frac{P(t)}{K}\right)^{-1} = ae^{rt}$$

for some positive constant a. We now need to solve this equation for $P(t)$:

$$
\begin{aligned}
ae^{rt} = P(t)\left(1 - \frac{P(t)}{K}\right)^{-1} &= \frac{KP(t)}{K - P(t)} \\
&\Rightarrow ae^{rt}(K - P(t)) = KP(t) \\
&\Rightarrow aKe^{rt} = ae^{rt}P(t) + KP(t) = (ae^{rt} + K)P(t) \\
&\Rightarrow P(t) = \frac{aKe^{rt}}{ae^{rt} + K} \\
&\Rightarrow P(t) = \frac{aK}{a + Ke^{-rt}}.
\end{aligned}
$$

We would like to express the arbitrary constant a in terms of the initial population:

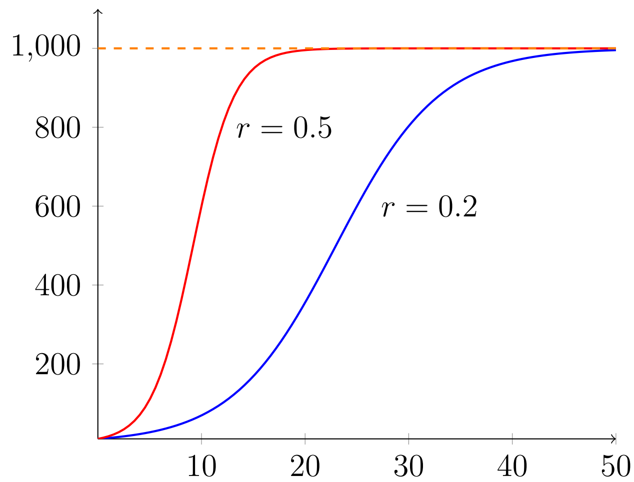
$$
\begin{aligned}
&P(0) = \frac{aKe^0}{ae^0 + K} = \frac{aK}{a + K} \\
&\Rightarrow P(0)(a + K) = aK \\
&\Rightarrow P(0)K = aK - P(0)a = a(K - P(0)) \\
&a = \frac{P(0)K}{K - P(0)}.
\end{aligned}
$$

Substituting this expression for a and simplifying gives the final form for the solution

$$
\boxed{P(t) = \frac{P(0)K}{P(0) + (K - P(0))e^{-rt}}.}
$$

(**Exercise:** How would things change if $P > K$?) It's easy to see from this equation that the limiting population is

$$
\lim_{t\to\infty} P(t) = K.
$$



Graph of $P(t)$ with $K = 1000$ and $P(0) = 10$ and two different growth rates: $r = 0.5$ in red and $r = 0.2$ in blue.

Exercise. A state game commission releases 40 elk into a game refuge. After 5 years, the elk population is 104. The commission believes that the refuge can support no more than 4000 elk. Use a logistic model to predict the elk population in 15 years.

SOLUTION: The carrying capacity is $K = 4000$, so the logistic model in this situation is

$$P'(t) = rP(t)\left(1 - \frac{P(t)}{4000}\right)$$

where we can determine r from the additional information we're given. The initial population size is $P(0) = 40$. From the solution to the logistic equation we derived above, we have

$$\begin{aligned} P(t) &= \frac{4000P(0)}{P(0) + (4000 - P(0))e^{-rt}} \\ &= \frac{160000}{40 + 3960e^{-rt}} \\ &= \frac{4000}{1 + 99e^{-rt}} \end{aligned}$$

We are given that $P(5) = 104$. Therefore,

$$104 = P(5) = \frac{4000}{1 + 99e^{-5r}}.$$

Solve for r:

$$\begin{aligned} 104 = \frac{4000}{1+99e^{-5r}} \quad &\Rightarrow \quad 104(1+99e^{-5r}) = 4000 \\ &\Rightarrow \quad e^{-5r} = \frac{1}{99}\left(\frac{4000}{104} - 1\right) = \frac{487}{1287} \\ &\Rightarrow \quad -5r = \ln\left(\frac{487}{1287}\right) \\ &\Rightarrow \quad r \approx 0.194. \end{aligned}$$

So our model for this population is

$$P(t) = \frac{4000}{1+99e^{-0.194\,t}}$$

So we would predict the population after 15 years to be

$$P(15) = \frac{4000}{1+99e^{-0.194\cdot 15}} \approx 626.$$

1. B. Separable—homogeneity trick.

An equation of the form

$$y' = F\left(\frac{y}{t}\right)$$

can be turned into a separable equation using the following substitution: let $v = y/t$. It follows that $y = vt$, and thus, $y' = v + tv'$ by the product rule. Then,

$$\begin{aligned} y' = F\left(\frac{y}{t}\right) \quad &\Rightarrow \quad v + tv' = F(v) \\ &\Rightarrow \quad \frac{v'}{F(v)-v} = \frac{1}{t} \\ &\Rightarrow \quad \int \frac{dv}{F(v)-v} = \int \frac{dt}{t}. \end{aligned}$$

Example. Solve

$$y' = \frac{y^2 + 2yt}{t^2}.$$

Notice that in the fraction on the right, the degree of every term in the numerator and denominator is 2. That's a sign of homogeneity. In fact, we have

$$\frac{y^2+2yt}{t^2} = \frac{y^2}{t^2} + \frac{2yt}{t^2} = \left(\frac{y}{t}\right)^2 + 2\left(\frac{y}{t}\right).$$

Substitute $v = y/t$ and $y' = v + tv'$ to transform the original equation into:

$$v + tv' = v^2 + 2v.$$

Separate variables and integrate. For convenience, we will assume that $y > 0$ and $t > 0$, and hence $v > 0$. Other cases can be handled similarly:

$$\frac{v'}{v^2 + v} = \frac{1}{t} \quad \Rightarrow \quad \int \frac{dv}{v^2 + v} = \int \frac{dt}{t}.$$

To integrate the left-hand side, use partial fractions:

$$\begin{aligned} \int \frac{dv}{v^2 + v} = \int \frac{dv}{v(v+1)} &= \int \left(\frac{1}{v} - \frac{1}{v+1} \right) dv \\ &= \ln(v) - \ln(v+1) + \tilde{c} \\ &= \ln \left(\frac{v}{v+1} \right) + \tilde{c}. \end{aligned}$$

We have found that

$$\ln \left(\frac{v}{v+1} \right) = \ln(t) + c.$$

Exponentiate and solve for v:

$$\frac{v}{v+1} = at. \quad \Rightarrow \quad v = \frac{at}{1 - at}.$$

Since $v = y/t$, we get

$$y = \frac{at^2}{1 - at}.$$

Considering an initial condition at $t = 0$ doesn't make much sense (why?). Let's write our solution in terms of an initial condition $I = y(1)$:

$$I = y(1) = \frac{a}{1 - a} \quad \Rightarrow \quad a = \frac{I}{1 + I}.$$

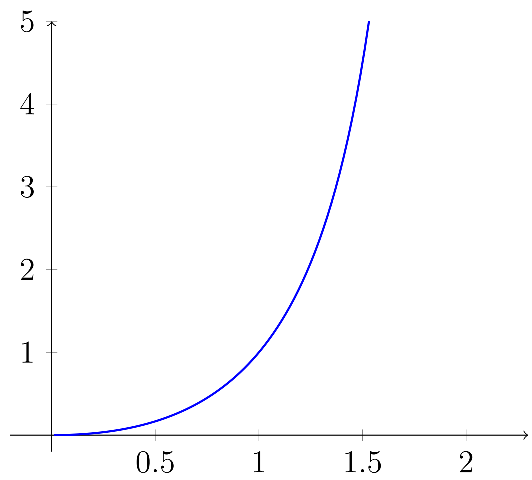
Substituting gives

$$y = \frac{I\, t^2}{I + 1 - It}.$$

For instance, if $I = 1$, we get the solution

$$y = \frac{t^2}{2 - t},$$

which is defined on the open interval $(-\infty, 2)$, however, recall that at some point along the way, we assumed $t > 0$. And, in fact, our original equation is undefined at $t = 0$. So the appropriate interval for this solution is $(0, 2)$:



Solution to $y' = (y^2 + yt)/t^2$ with $y(1) = 1$.

Here is the Sage code for solving this equation:

```
sage: t = var('t')
sage: y = function('y')(t)
sage: desolve(diff(y,t)-(y^2+2*y*t)/t^2,y)
-(t^2 + t*y(t))/y(t) &=  _C
sage: desolve(diff(y,t)-(y^2+2*y*t)/t^2,y,ics=[1,1])
-(t^2 + t*y(t))/y(t) &=  -2
```

For the second call to desolve, I've included initial conditions, y first and t second:

```
 ics = [y(t_0), t_0].
```

Week 1, Friday: Exact equations. Integrating factors

II. A. Exact equations.

An exact differential equation has the form

$$M(t,y) + N(t,y)\,\frac{dy}{dt} = 0.$$

where

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}.$$

We would like to find a solution that defines y implicitly, i.e., we are looking for a function of the form

$$\Phi(t,y) = 0.$$

If we had such a function, then by the chain rule,

$$0 = \frac{d\Phi}{dt} = \frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial y}\frac{dy}{dt}.$$

Then Φ would be a solution if

$$M(t,y) = \frac{\partial \Phi}{\partial t} \quad \text{and} \quad N(t,y) = \frac{\partial \Phi}{\partial y}.$$

Note that the conditions on the partials of M and N which are required of an exact equation would then follow necessarily:

$$\frac{\partial M}{\partial y} = \frac{\partial^2 \Phi}{\partial t\,\partial y} = \frac{\partial N}{\partial t}.$$

The trick then is to reverse-engineer this argument. Since $M(t,y) = \dfrac{\partial \Phi}{\partial t}$, we integrate M with respect to t:

$$\Phi(t,y) = \int M(t,y)\,dt =: m(t,y) + f(y)$$

where f is an arbitrary function of y. Then we use the fact that $N(t,y) = \dfrac{\partial \Phi}{\partial y}$ to determine $f(y)$:

$$N(t,y) = \frac{\partial \Phi}{\partial y} = \frac{\partial}{\partial y}(m(t,y) + f(y)).$$

This determines $f(y)$ up to a constant.

Note for those who have seen differential forms: Recall that the differential form ω is *exact* if there is a form ψ such that $d\psi = \omega$. Since $d^2 = 0$, such forms are automatically *closed*: $d\omega = d^2\psi = 0$. In our case, we are considering the 0-form, $\psi = \Phi(t,y)$, and then

$$d\psi = \frac{\partial \Phi}{\partial t}\, dt + \frac{\partial \Phi}{\partial y}\, dy. = M(t,y)\, dt + N(t,y)\, dy.$$

Another way of saying the same thing is that the vector field

$$(t,y) \mapsto (M(t,y), N(t,y))$$

is the gradient vector field $\nabla \Phi$.

Example. Solve

$$\sin(t+y) + (2y + \sin(t+y))y' = 0.$$

The equation is not separable. However, it is exact since

$$\frac{\partial}{\partial y} \sin(t+y) = \cos(t+y) = \frac{\partial}{\partial t}(2y + \sin(t+y)).$$

We have $M(t,y) = \sin(t+y)$ and $N(t,y) = 2y + \sin(t+y)$. To solve the equation, note that

$$\int M(t,y)\, dt = -\cos(t+y) + f(y)$$

for some $f(y)$, and then

$$\frac{\partial}{\partial y}(-\cos(t+y) + f(y)) = N(t,y) = 2y + \sin(t+y)$$

implies that

$$\frac{df}{dy} = 2y.$$

Hence, $f(y) = y^2 + \tilde{c}$. Our final solution is

$$-\cos(t+y) + y^2 = c.$$

Slope fields. Let $y = y(t)$ be the solution to a differential equation $y' = F(y, t)$. The graph of $y(t)$ is a curve. At time t_0, the curve passes through the point $(t_0, y(t_0))$ and has slope $y'(t_0) = F(t_0, y(t_0))$. Imagine attaching to each point $(a, b) \in \mathbb{R}^2$ a tiny line segment with slope $F(a, y(a))$. Any solution curve will then be tangent to each line segment it meets. (There will be lots of solutions, depending on the initial condition.) For example, Figure 3.1 creates the slope field and exhibits several possible solutions. Here is the Sage code used to produce the figure:

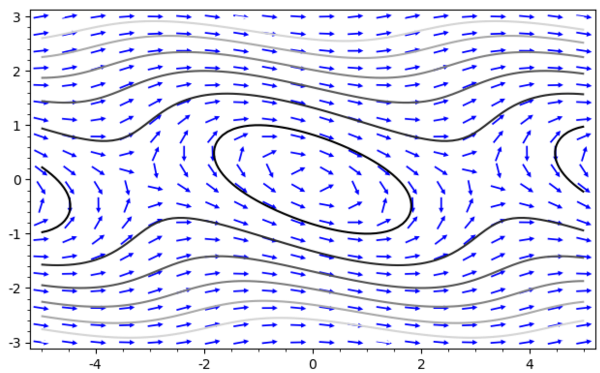


Figure 3.1: Slope field and solutions for $\sin(t + y) + (2y + \sin(t + y))y' = 0$.

```
sage: v = plot_slope_field(-sin(t+y)/(2*y+sin(t+y)),(t,-5,5),(y,-3,3),
...: headaxislength=3, headlength=3,color='blue')
sage: c = contour_plot(-cos(t+y)+y^2,(t,-5,5),(y,-3,3),fill=false)
sage: v + c
Launched png viewer for Graphics object consisting of 2 graphics primitives
```

II. B. Exact after multiplying through by integrating factor.

We are again interested in solving

$$M(t, y) + N(t, y)\,\frac{dy}{dt} = 0,$$

but this time, we don't assume that $\partial M/\partial y = \partial N/\partial t$. In that case, we look for a function $\mu(t, y)$ such that

$$\mu(t, y)M(t, y) + \mu(t, y)N(t, y)\,\frac{dy}{dt} = 0,$$

is exact. In fact, μ always exists:

Proof. Let Φ be such that $\Phi(t, y) = 0$ (we can talk about the existence of Φ later, but for now let's assume it exists). Differentiate with respect to t, as before, and use the chain rule

$$0 = \frac{d\Phi}{dt} = \frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial y}\frac{dy}{dt}.$$

We have

$$\frac{dy}{dt} = -\frac{\partial \Phi/\partial t}{\partial \Phi/\partial y} = -\frac{M(t,y)}{N(t,y)},$$

and, hence,

$$\frac{\partial \Phi/\partial t}{M(t,y)} = \frac{\partial \Phi/\partial y}{N(t,y)} =: \mu(t,y),$$

where we have just now defined μ. It follows that

$$0 = \mu(t,y)M(t,y) + \mu(t,y)N(t,y)\frac{dy}{dt} = \frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial y}\frac{dy}{dt},$$

which is now exact. $\square$

That's the good news. The bad news is that it might not be easy to find μ. I typical strategy is to assume that μ has a certain form involving parameters, and then try to figure out what values for the parameters will make your equation exact.

Example. Solve

$$ty^2 + 4t^2y + (3t^2y + 4t^3)\frac{dy}{dt} = 0.$$

This equation is not exact. We'll try to find an integrating factor of the form $\mu(t, y) = t^m y^n$. So we would like for

$$(t^m y^n)(ty^2 + 4t^2y) + t^m y^n (3t^2y + 4t^3)\frac{dy}{dt} = 0$$

to be exact. We need

$$\frac{\partial}{\partial y}(t^{m+1}y^{n+2} + 4t^{m+2}y^{n+1}) = \frac{\partial}{\partial t}(3t^{m+2}y^{n+1} + 4t^{m+3}y^n).$$

In other words, we need

$$(n+2)t^{m+1}y^{n+1} + 4(n+1)t^{m+2}y^n = 3(m+2)t^{m+1}y^{n+1} + 4(m+3)t^{m+2}y^n.$$

Equate coefficients:

$$n + 2 = 3(m+2) \quad \text{and} \quad 4(n+1) = 4(m+3).$$

Solving this system of linear equations yields $m = -1$ and $n = 1$. Our integrating factor is $\mu(t,y) = y/t$. Ah, ha! That reminds me of the homogeneity trick. In fact, solving for dy/dt in the original equation does give the form $y' = F(y/t)$! So we could have solved this with our earlier machinery. Nevertheless, we'll continue from here. Multiplying through by the integrating factor transforms our original equation into

$$y^3 + 4ty^2 + (3ty^2 + 4t^2y)\frac{dy}{dt} = 0,$$

which is now exact with

$$M = y^3 + 4ty^2 \quad \text{and} \quad N = 3ty^2 + 4t^2y.$$

(Check that $\partial M/\partial y = \partial N/\partial t$ to be sure.) Solve the exact equation:

$$\Phi(t,y) = \int M\,dt = ty^3 + 2t^2y^2 + f(y)$$

implies

$$N(t,y) = \frac{\partial \Phi}{\partial y} = 3ty^2 + 4t^2y + \frac{df}{dy}.$$

Comparing with $N(t,y)$ shows that $df/dy = 0$. Hence, $f(y) = \tilde{c}$, a constant. Our solution:

$$ty^3 + 2t^2y^2 = c,$$

(where $c = -\tilde{c}$, is just another constant). Figure 3.2 give the slope field and several solutions. Figure 3.3 plots the function $z = ty^3 + 2t^2y^2$. The level sets of this function are solutions to the differential equation.

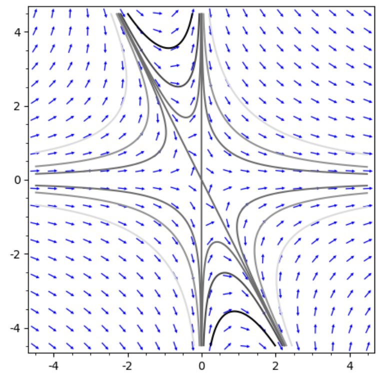


Figure 3.2: Slope field and solutions to $ty^2 + 4t^2y + (3t^2y + 4t^3)\frac{dy}{dt} = 0$.

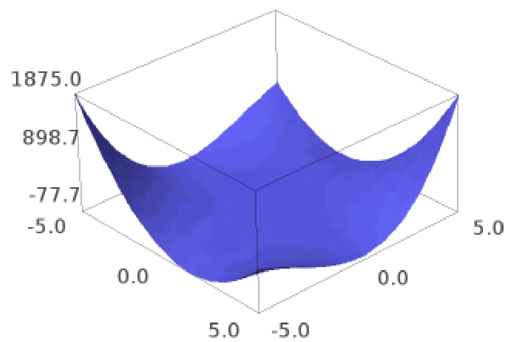


Figure 3.3: Plot of the surface $z = ty^3 + 2t^2y^2$.

Week 2, Monday: First-order linear. Linear homogeneous constant coefficients

III. A. First-order linear.

A first-order linear equation has the form

$$\frac{dy}{dt} + p(t)y = q(t).$$

It is solved using the integrating factor $e^{\int p(t)\,dt}$: multiplying the equation through by this factor gives

$$e^{\int p(t)\,dt)} \left(\frac{dy}{dt} + p(t)y \right) = e^{\int p(t)\,dt} q(t). \tag{4.1}$$

By the chain rule and the fundamental theorem of calculus, the left-hand side of this equation is

$$\frac{d}{dt}\left(e^{\int p(t)\,dt} y \right).$$

So we can integrate equation (4.1) to get

$$e^{\int p(t)\,dt} y = \int e^{\int p(t)\,dt} q(t)\,dt,$$

and then solve for y.

Example. Consider the following equation

$$\cos(t)\,y' + y = \sin(t)$$

with initial condition $y(0) = 1$. Dividing by $\cos(t)$ puts the equation into standard form (note that $\cos(t) \neq 0$ near $t = 0$):

$$y' + \sec(t)\,y = \tan(t).$$

The integrating factor is

$$e^{\int \sec(t)\,dt} = e^{\ln(\sec(t)+\tan(t))} = \sec(t) + \tan(t).$$

(Near $t = 0$, we have $\sec(t) + \tan(t) > 0$. Multiplying the equation through by the integrating factor gives

$$(\sec(t) + \tan(t))\, y' + (\sec^2(t) + \sec(t)\tan(t))\, y = (\sec(t) + \tan(t))\tan(t).$$

Integrate both sides:

$$\begin{aligned}
(\sec(t) + \tan(t))y &= \int (\sec(t) + \tan(t))\tan(t)\,dt \\
&= \int (\sec(t)\tan(t) + \tan^2(t))\,dt \\
&= \int \sec(t)\tan(t)\,dt + \int \tan^2(t))\,dt \\
&= \sec(t) + \int \tan^2(t))\,dt \\
&= \sec(t) + \int \left(\sec^2(t) - 1\right)\,dt \\
&= \sec(t) + \tan(t) - t + c
\end{aligned}$$

Therefore,

$$y = \frac{\sec(t) + \tan(t) - t + c}{\sec(t) + \tan(t)}.$$

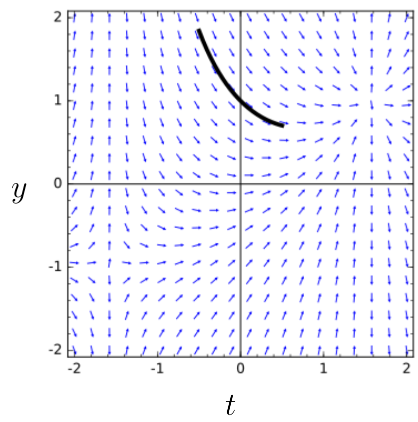
Let's write this in terms of the initial condition:

$$1 = y(0) = \frac{\sec(0) + \tan(0) + c}{\sec(0) + \tan(0)} = 1 + c.$$

So $c = 0$, and the solution is

$$y = \frac{\sec(t) + \tan(t) - t}{\sec(t) + \tan(t)} = 1 - \frac{t}{\sec(t) + \tan(t)}.$$

Here is a plot of the slope field and our solution:



Note the weirdness around $t = \pi/2$, where $\cos(t)$ is 0. It's exactly where we divide by zero in our calculations where the interesting stuff happens.

III. B. Bernoulli-type first-order linear.

We are now interested in solving an equation of the form

$$\frac{dy}{dt} + p(t)y = q(t)y^m,$$

where $m \neq 1$. The trick here is to reduce the equation to a standard first-order linear equation with the substitution $u = y^{1-m}$. In that case, we have

$$u' = (1-m)y^{-m}y'.$$

Multiply the original equation through by $(1-m)y^{-m}$

$$(1-m)y^{-m}y' + (1-m)p(t)y^{1-m} = (1-m)q(t)$$

and substitute:

$$u' + (1-m)p(t)u = (1-m)q(t).$$

Example. Consider the equation

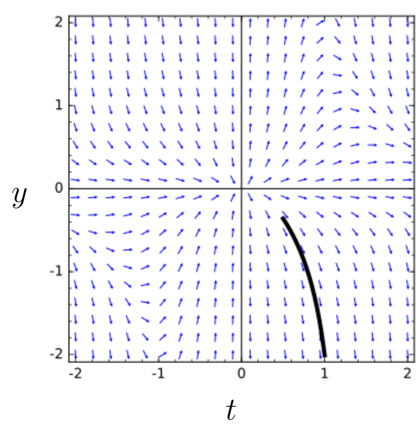
$$y' = \frac{2y}{t} - t^2y^2$$

with initial condition $y(1) = -2$. This is Bernoulli-type with $m = 2$, so we make the substitution $u = y^{-1}$. This transforms the equation into the first-order linear equation

$$u' + \frac{2u}{t} = t^2.$$

The integrating factor is
$$e^{\int (2/t)\,dt} = t^2.$$
Multiply through by it and integrate:
$$\begin{aligned} t^2u' + 2tu = t^4 \quad &\Rightarrow \quad \frac{d}{dt}(t^2u) = t^4 \\ &\Rightarrow \quad t^2u = \int t^4\,dt = \frac{1}{5}t^5 + c \\ &\Rightarrow \quad \frac{t^2}{y} = \frac{1}{5}t^5 + c \end{aligned}$$

The initial condition gives us:
$$-\frac{1}{2} = \frac{1}{5} + c \quad \Rightarrow \quad c = -\frac{7}{10}.$$
The solution is
$$\frac{t^2}{y} = \frac{1}{5}t^5 - \frac{7}{10} \quad \Rightarrow \quad y = \frac{10t^2}{2t^5 - 7}.$$
The slope field and our solution:



IV. A. Linear homogeneous constant coefficients (LHCC).

We are now interested in solving a differential equation of the form
$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0$$

where $y^{(i)}$ denotes the i-th derivative of y with respect to t. The a_i are constants. The word "homogeneous" refers to the fact that a 0 appears to the right of the equals sign. Letting $D := d/dt$, we can write the above equation as

$$(D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0)y = 0$$

or just

$$P(D)y = 0$$

where P is the polynomial $P(x) = \sum_{i=0}^{n} a_i x^i$.

Main theory.

1. The solution space is linear: suppose that y_1 and y_2 are solutions, i.e., $P(D)y_1 = P(D)y_2 = 0$. Let α be a constant. Then

$$P(D)(y_1 + \alpha y_2) = P(D)y_1 + \alpha\, P(D)y_2 = 0$$

   by linearity of differentiation.

2. The "basic" solutions have the form e^{rt} (more on this later).

3. When determining which values for r are suitable, something nice happens:

$$P(D)e^{rt} = \sum_{i=0}^{n} a_i D^i e^{rt} = \sum_{i=0}^{n} a_i r^i e^{rt} = P(r)e^{rt}.$$

   Since $e^{rt} > 0$, we get a solution $P(D)e^{rt} = 0$ if and only if $P(r) = 0$. So the values for r that give solutions are exactly the zeros of the polynomial P. The polynomial $P(r)$ is called *characteristic polynomial* for the equation.

4. For uniqueness, we specify $y(t_0), \ldots, y^{(n-1)}(t_0)$.

Example. Solve

$$y'' - y' - 6y = 0$$

with initial conditions $y(0) = 0$ and $y'(0) = 1$.

SOLUTION: Find the zeros of the characteristic polynomial:

$$P(r) = r^2 - r - 6 = (r+2)(r-3) = 0 \quad \Leftrightarrow \quad r = -2, 3.$$

The general solution is

$$y = ae^{-2t} + be^{3t}.$$

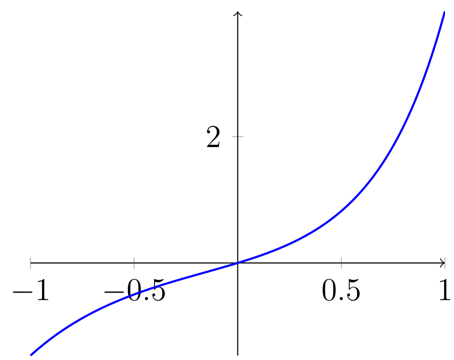
To satisfy the initial conditions, we need

$$\begin{aligned} a + b &= 0 \\ -2a + 3b &= 1. \end{aligned}$$

Solving this system gives $a = -1/5$ and $b = 1/5$. So the solution is

$$y = -\frac{1}{5}\, e^{-2t} + \frac{1}{5}\, e^{3t}.$$

A graph of the solution:



Week 2, Wednesday: Bernoulli equation. LHCC: complex roots and repeated roots. Method of undetermined coefficients

Aside on Bernoulli-type equations. Imagine a moving particle with velocity v and a force $F = F(v)$ acting on the particle against its direction of motion—a frictional force. It is reasonable to assume $F(-v) = -F(v)$. Now suppose that F has a power series expansion

$$F(v) = a_0 + a_1 v + a_2 v^2 + \dots.$$

The fact that $F(-v) = -F(v)$ implies that the even terms vanish:

$$F(v) = a_1 v + a_3 v^3 + a_5 v^5 + \dots$$

As a first approximation, we could take

$$F(v) = a_1 v$$

Since force is proportional to acceleration, i.e., $F(v) = \text{constant} \cdot v'$, we can write this model of friction as

$$v' = \alpha v.$$

The solution is $v = e^{\alpha t}$, and for our purposes, we take $\alpha < 0$. The next best approximation is to use the first two terms of the series:

$$v' = \alpha v + \beta v^3,$$

which is a Bernoulli-type equation. Question: what is the behavior of a particle whose motion is governed by this equation?

LHCC. We now continue our discussion of linear homogeneous constant coefficients equations. These have the form

$$y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0,$$

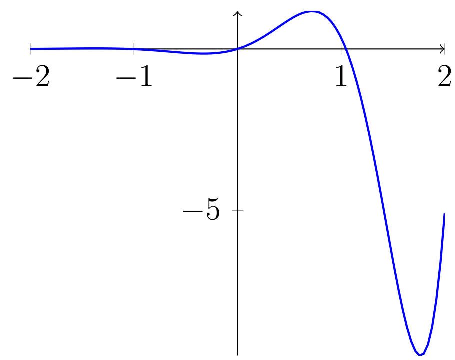
or, more succinctly,

$$P(D)y = 0$$

where $D = d/dt$ and $P(x) = \sum_{i=0}^{n} a_i x^i$. The trick is to look for solutions of the form $y = e^{rt}$. We have $P(D)e^{rt} = P(r)e^{rt}$. So we have a solution of that form exactly for the zeros of P.

Example. Solve
$$y'' - 4y' + 13y = 0$$
with initial conditions $y(0) = 0$ and $y'(0) = 1$.

SOLUTION: Find the zeroes of the characteristic polynomial (quadratic equation to the rescue!):
$$r^2 - 4r + 13 = 0 \quad \Rightarrow \quad r = 2 \pm 3i.$$
So the general solution is
$$y = Ae^{(2+3i)t} + Be^{(2-3i)t}.$$
We would like to express the solution in terms of real numbers:
$$\begin{aligned} y &= Ae^{(2+3i)t} + Be^{(2-3i)t} \\ &= Ae^{2t}(\cos(3t) + i\sin(3t)) + Be^{2t}(\cos(3t) - i\sin(3t)) \\ &= (A+B)e^{2t}\cos(3t) + (A-B)ie^{2t}\sin(3t)) \\ &= ae^{2t}\cos(3t) + be^{2t}\sin(3t). \end{aligned}$$
The general real solution is
$$y = ae^{2t}\cos(3t) + be^{2t}\sin(3t).$$
Now we handle the initial conditions:
$$0 = y(0) = ae^0\cos(0) + be^0\sin(0) = a.$$
So $y = be^{2t}\sin(3t)$. Then
$$1 = y' = 3be^0\cos(0) + 2be^0\sin(0) = 3b.$$
So the solution is
$$y = \frac{1}{3}e^{2t}\sin(3t)$$
Graph of solution:



There is one final wrinkle in the story: what if $P(r)$ has a repeated root? Say $P(r)$ has a factor of the form $(r-\lambda)^k$. In that case, the general solution will include $a_0e^{\lambda t} + a_1te^{\lambda t} + \cdots + a_kt^{k-1}e^{\lambda t}$. We will be able to understand why this is the case once we move to the higher-dimensional linear theory. For now, you're invited to check that $(D-\lambda)^k t^\ell e^{\lambda t} = 0$ for $0 \le \ell \le k-1$ by hand. That way, you'll at least see these are solutions.

Examples.

1. Consider the equation
$$y''' + 6y'' + 12y' + 8y = 0.$$
Its characteristic polynomial is
$$P(r) = r^3 + 6r^2 + 12r + 8 = (r+2)^3.$$
So $P(r)$ as the root $r = -2$ of multiplicity 3. The general solution to the equation is therefore
$$y = ae^{-2t} + bte^{-2t} + ct^2e^{-2t} = (a + bt + ct^2)e^{-2t}.$$

2. Consider the equation
$$y^{(5)} + 3y^{(4)} + 3y^{(3)} + y^{(2)} = 0.$$
Its characteristic polynomial is
$$P(r) = r^5 + 3r^4 + 3r^3 + r^2 = r^2(r+1)^3.$$
The roots are $r = 0$ with multiplicity 2 and $r = -1$ with multiplicity 3. Notice that the root $r = 0$ will correspond to solutions involving $e^{0\cdot t} = 1$. The general solution is
$$a_1 + a_2t + a_3e^{-t} + a_4te^{-t} + a_5t^2e^{-t}.$$

3. Say we are considering a LHCC differential equation with characteristic polynomial

$$P(r) = r^3(r-2)^2(r^2+9)^2 = 0.$$

The roots are $r = 0, 2, \pm 3i$ with multiplicities $3, 2, 2$, respectively. The general solution is

$$y = a_1 + a_2 t + a_3 t^2 + b_1 e^{2t} + b_2 t e^{2t} + c_1 \cos(3t) + c_2 \sin(3t) + c_3 t \cos(3t) + c_4 t \sin(3t).$$

V. Method of undetermined coefficients.

We now consider inhomogeneous linear equations with constant coefficients. These have the form

$$P(D)y = f(t).$$

Where P is a polynomial and $D = d/dt$, as before. To solve this equation, we first try to find a particular solution y_p. We then find a general solution y_h to $P(D)y = 0$, the associated homogeneous system. The general solution to the inhomogeneous system is then $y_h + y_p$. The new challenge here is to find the particular solution, y_p. The idea we will use is to guess the form of y_p and adjust parameters. Here is a table that may be of help ("poly" means "polynomial"):

$f(t)$	guess
polynomial	general polynomial of some degree
e^{rt}	ae^{rt}
(poly)e^{rt}	(general poly)e^{rt}
$\cos(\omega t)$ or $\sin(\omega t)$	$a\cos(\omega t) + b\sin(\omega t)$
(poly)$e^{rt}\cos(\omega t)$ or (poly)$e^{rt}\sin(\omega t)$	(gen poly)$e^{rt}\cos(\omega t)$ + (gen poly)$e^{rt}\sin(\omega t)$

Example. Consider the equation

$$y'' - 2y' + y = t^2.$$

We guess a particular equation of the form

$$y = a_0 + a_1 t + a_2 t^2.$$

In that case, we have

$$\begin{aligned} y'' - 2y' + y &= 2a_2 - 2(a_1 + 2a_2 t) + (a_0 + a_1 t + a_2 t^2) \\ &= (2a_2 - 2a_1 + a_0) + (-4a_2 + a_1)t + a_2 t^2. \end{aligned}$$

Set this equal to t^2 and compare coefficients:

$$\begin{aligned} 0 &= 2a_2 - 2a_1 + a_0 \\ 0 &= -4a_2 + a_1 \\ 1 &= a_2. \end{aligned}$$

Solving the system gives

$$a_0 = 6,\ a_1 = 4,\ a_2 = 1.$$

So a particular solution is

$$y_p = 6 + 4t + t^2.$$

(Check!) We now solve the associated homogeneous equation

$$y'' - 2y + y = 0.$$

The characteristic polynomial is

$$r^2 - 2r + 1 = (r-1)^2,$$

which has the zero $r = 1$ with multiplicity 2. So the general solution to the homogeneous system is

$$y_h = ae^t + bte^t.$$

The most general solution to the original equation is then

$$y = y_h + y_p = ae^t + bte^t + 6 + 4t + t^2.$$

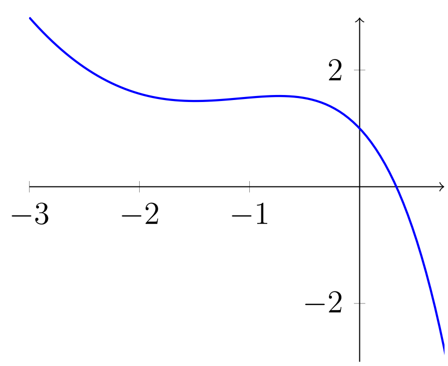
Suppose we are given initial conditions $y(0) = 1$ and $y'(0) = -2$. Then

$$\begin{aligned} 1 &= y(0) = a + 6 \\ -2 &= y'(0) = a + b + 4. \end{aligned}$$

Therefore, $a = -5$ and $b = -1$. The solution is

$$y = -5e^t - te^t + 6 + 4t + t^2.$$

Graph of solution:



Week 2, Friday: Special second-order equations

V. Method of undetermined coefficients.

We look at one more example of the method of undetermined coefficients. Consider the equation

$$y'' - 2y' + y = t\cos(3t).$$

We guess a particular solution of the form

$$y = (a_0 + a_1 t)\cos(3t) + (b_0 + b_1 t)\sin(3t).$$

Then

$$\begin{aligned} y' &= (a_1 + 3b_0 + 3b_1 t)\cos(3t) + (-3a_0 + b_1 - 3a_1 t)\sin(3t) \\ y'' &= (-9a_0 + 6b_1 - 9a_1 t)\cos(3t) + (-6a_1 - 9b_0 - 9b_1 t)\sin(3t) \end{aligned}$$

So we have

$$\begin{aligned} y'' - 2y' + y = (&-8a_0 - 2a_1 - 6b_0 + 6b_1 - (8a_1 + 6b_1)t)\cos(3t) \\ &+ (6a_0 - 6a_1 - 8b_0 - 2b_1 + (6a_1 - 8b_1)t)\sin(3t) \end{aligned}$$

Set this equal to $t\cos(3t)$ and compare coefficients to get the system on linear equations

$$\begin{aligned} 0 &= -8a_0 - 2a_1 - 6b_0 + 6b_1 \\ 1 &= -8a_1 - 6b_1 \\ 0 &= 6a_0 - 6a_1 - 8b_0 - 2b_1 \\ 0 &= 6a_1 - 8b_1 \end{aligned}$$

Solving this system gives the particular solution

$$y_p = -\frac{1}{250}\left(13 + 20\,t\right)\cos\left(3\,t\right) - \frac{3}{250}\left(-3 + 5\,t\right)\sin\left(3\,t\right).$$

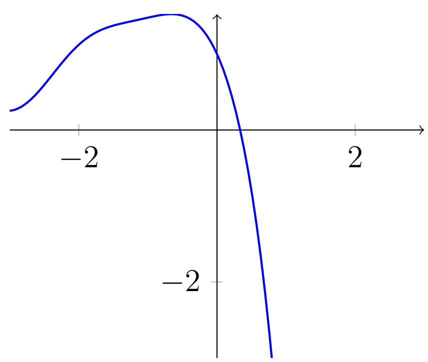
The corresponding homogeneous equation, $y'' - 2y' + y = 0$, has a general solution $ae^t + bte^t$. So the general solution to our inhomogeneous equation is

$$y = ae^t + bte^t - \frac{1}{250}\,(13 + 20\,t)\cos(3\,t) - \frac{3}{250}\,(-3 + 5\,t)\sin(3\,t)$$

Let's again consider the initial conditions $y(0) = 1$ and $y'(0) = -2$. Plugging these into the general solution and its derivative allow us to determine a and b. The result is

$$y = \frac{263}{250}\,e^t - \frac{77}{25}\,te^t - \frac{1}{250}\,(13 + 20\,t)\cos(3\,t) - \frac{3}{250}\,(-3 + 5\,t)\sin(3\,t)\,.$$

Graph of solution:



VI. A. Second-order. Given a second-order equation of the form

$$H(t, y', y'') = 0$$

i.e., missing a y-term, we can reduce the order of the equation with the substitution $v = y'$.

Example. Consider the equation

$$ty'' + 4y' = t^2.$$

Substitute $v = y'$ to get the equation

$$tv' + 4v = t^2.$$

If $t \neq 0$, this becomes the standard first-order equation

$$v' + \frac{4}{t}\,v = t.$$

Say $t > 0$. Then the integrating factor is $\exp\left(\int \frac{4}{t}\,dt\right) = t^4$. Multiplying through (and using the product rule), we have

$$t^4\,v' + 4t^3\,v = (t^4 v)' = t^5.$$

Integrate:

$$t^4 v = \frac{1}{6}t^6 + c.$$

Now substitute back $v = y'$:

$$t^4 y' = \frac{1}{6}t^6 + c.$$

This is separable:

$$\begin{aligned} y' = \frac{1}{6}t^2 + \frac{c}{t^4} \quad \Rightarrow \quad y &= \frac{1}{18}t^3 - \frac{1}{3}\cdot\frac{c}{t^3} + b \\ &= \frac{1}{18}t^3 + \frac{a}{t^3} + b. \end{aligned}$$

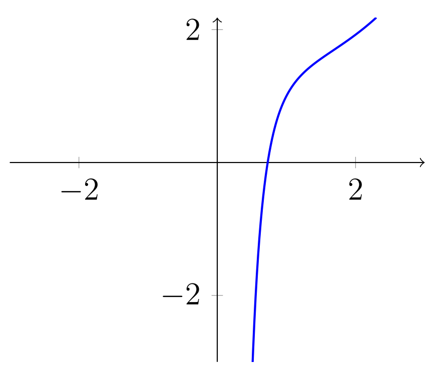
Suppose the initial conditions are $y(1) = 1$ and $y'(1) = 2$. Then

$$\begin{aligned} 1 &= \frac{1}{18} + a + b \\ 2 &= \frac{1}{6} - 3a, \end{aligned}$$

which implies $a = -11/18$ and $b = 14/9$. The solution is

$$y = \frac{1}{18}t^3 - \frac{11}{18}\frac{1}{t^3} + \frac{14}{9}.$$

Graph of solution:



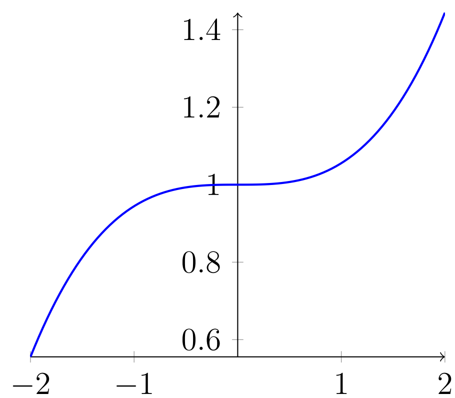
Solutions defined near $t = 0$? Our method of forcing the equation into the form of a standard first-order equation requires dividing by t, and hence, assumes that $t \neq 0$. What if we really want a solution defined near $t = 0$? My approach was to suppose the solution can be expanded in terms of a power series $y = a_0 + a_1 t + a_2 t^2 + \dots$ Plug this series into the equation $ty'' + 4y'$ and set the result equal to t^2. Now compare coefficients and hope we can solve for the a_i. If you think about it, we only need to consider series where $a_i = 0$ for $i \geq 4$. So assume $y = a_0 + a_1 t + a_2 t^2 + a_3 t^3$. We have

$$\begin{aligned} ty'' + 4y' &= t(2a_2 + 6a_3 t) + 4(a_1 + 2a_2 t + 3a_3 t^2) \\ &= 4a_1 + 10a_2 t + 18a_3 t^2. \end{aligned}$$

Setting the result equal to t^2 and comparing coefficients gives $a_1 = a_2 = 0$, and $a_3 = 1/18$. The solution is

$$y = a_0 + \frac{1}{18} t^3.$$

Graph of solution with initial condition $y(0) = 1$:



Note that the only possibly initial condition for $y'(0)$ is $y'(0) = 0$ (why?). Since this is a second-order equation, we'd expect to be able to set initial conditions for both y and y'. We should try to remember to come back to this example when we talk about existence and uniqueness of solutions.

VI. B. Second-order equation.

Given a second-order equation of the form

$$H(y, y', y'') = 0$$

i.e., missing t, we again make the substitution $v = y'$, but then use the chain rule like so

$$y'' = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy}.$$

Substituting, our original equation becomes

$$H\left(y, v, v\frac{dv}{dy}\right) = 0.$$

After we find v as a function of y, we solve for y by integrating, as before.

Example. Consider the equation

$$y'' + (y')^3\, y = 0.$$

Let $v = y'$ and substitute as above to get

$$v\frac{dv}{dy} + v^3 y = 0.$$

This is first-order linear, but even better, it is separable. Supposing $v > 0$, the equation becomes

$$\frac{1}{v^2}\frac{dv}{dy} = -y.$$

Integrate:

$$\begin{aligned}
\int \frac{1}{v^2}\, dv = -\int y\, dy \quad &\Rightarrow \quad -\frac{1}{v} = -\frac{1}{2}y^2 + \tilde{c} \\
&\Rightarrow \quad v = \frac{2}{y^2 - 2\tilde{c}} \\
&\Rightarrow \quad v = \frac{2}{y^2 + c}.
\end{aligned}$$

Now substitute back in $v = y'$:

$$y' = \frac{2}{y^2 + c} \quad \Rightarrow \quad \int (y^2 + c)\, dy = 2\int\, dt \quad \Rightarrow \quad \frac{1}{3}y^3 + cy = 2t + b.$$

Suppose our initial conditions are $y(1) = 0$ and $y'(1) = 1$. Then

$$\frac{1}{2}\cdot 0^3 + c\cdot 0 = 2\cdot 1 + b \quad \Rightarrow \quad b = -2.$$

So the equation becomes

$$\frac{1}{3}y^3 + cy = -2 + 2t.$$

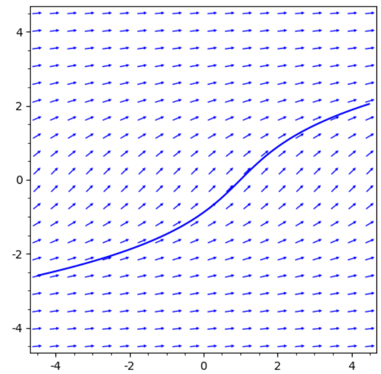
To use the second condition, take derivatives with respect to t:

$$y^2\, y' + cy' = 2.$$

Plug in $y(1) = 0$ and $y'(1) = 1$ to find $c = 2$. The solution, implicitly, is

$$\frac{1}{3}y^3 + 2y = -2 + 2t.$$

Here is a picture of the slope field and our solution:



VII. Duh.

If your method of solving a differential equation is not working due to a troublesome set of initial conditions, consider obvious/trivial solutions.

Example. We just solved the equation

$$y'' + (y')^3 y = 0.$$

for a particular set of initial conditions. If you look back at our method solution, you'll see that we can find a solution satisfying any initial conditions $y(t_0) = \alpha$ and $y'(t_0) = \beta$, except for those where $\beta = 0$. That's because we divided by $v = y'$ in the course of our solution. What do we do for the troublesome case of $\beta = 0$? Applying the "duh" method, we immediately find the solution $y = \alpha$, a constant function.

Challenge. Solve

$$y'' + (y')^3 y = t.$$

with initial condition $y(0) = 1$ and $y'(0) = 0$.

Week 3, Monday: Matrix exponentiation

Let $F = \mathbb{R}$ or $\mathbb{C}$, and let $M_n(F)$ denote $n \times n$ matrices with coefficients in F. The derivative of a curve $x(t) = (x_1(t), \ldots, x_n(t))$ in F^n with respect to t gives the curve's tangent direction or velocity at time t:

$$\dot{x} := x'(t) := \left(\frac{dx_1}{dt}, \ldots, \frac{dx_n}{dt} \right).$$

We are interested in finding x such that

$$x' = Ax$$

and satisfying some initial condition $x(0) = x_0 \in F^n$. If $n = 1$, then $A = a \in F$, and we have already seen the solution $x = x_0 e^{at} = e^{at} x_0$. It turns out that the solution in the case $n = 1$ is just a space case of the solution for $n \geq 1$:

$$x = e^{At} x_0. \tag{7.1}$$

Our first goal is to make sense of equation (7.1) (e.g., what does it mean to exponentiate a matrix?) and then prove that it is the unique solution.

Definition. A *norm* on a vector space V over F is a mapping

$$\| \ \| \colon V \to \mathbb{R}$$

satisfying

1. (positive definite) $\|v\| \geq 0$ for all $v \in V$, and $\|v\| = 0$ if and only if $v = 0$.
2. (absolute homogeneity) $\|\alpha v\| = |\alpha| \|v\|$ for all $v \in V$ and $\alpha \in F$.
3. (triangle inequality) $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$.

Examples. The usual absolute value on F^n is a norm. If $F = \mathbb{R}$, we have

$$\|x\| := |x| := \sqrt{x \cdot x} = \sqrt{\textstyle\sum_j x_j^2}$$

and if $F = \mathbb{C}$, we have

$$\|x\| := |x| = \sqrt{x \cdot \bar{x}} = \sqrt{\textstyle\sum_j |x_j|^2}.$$

Note: if $x_j = a_j + b_j i$ with $a_j, b_j \in \mathbb{R}$, then

$$\|x\| = |x| = \sqrt{\textstyle\sum_j (a_j^2 + b_j^2)},$$

which is the length of $x \in \mathbb{C}^n$ thought of as a vector in $\mathbb{R}^{2n}$. As indicated above, we use the usual absolute value notation, $|x|$ for this norm.

The case $n = 1$ says the usual absolute value on F is a norm on F.

Given a norm $\| \ \|$ on a vector space V, we can define a *metric* on V (i.e., a distance function) by

$$d(v, w) := \|v - w\|.$$

The following properties of this distance function are easy to verify:

1. (positive definite) $d(v, w) \geq 0$ for all $v, w \in V$, and $d(v, w) = 0$ if and only if $v = w$.
2. (symmetry) $d(v, w) = d(w, v)$ for all $v, w \in V$.
3. (triangle inequality) $d(u, w) \leq d(u, v) + d(v, w)$ for all $u, v, w \in V$.

The following proposition implies that two norms on a vector space will define the same topology ("sense of closeness") on that space:

Proposition. Let $\| \ \|_1$ and $\| \ \|_2$ be two norms on a finite-dimensional vector space V over F. Then these norms are *equivalent* in the following sense: there exist positive real numbers a, b such that

$$a\|v\|_2 \leq \|v\|_1 \leq b\|v\|_2$$

for all $v \in V$.

Sketch of proof.

Step 1. If the displayed set of inequalities holds, say $\| \ \|_1 \sim \| \ \|_2$. Prove that $\sim$ is an equivalence relation.

Step 2. By Step 1, it suffices to prove the result when $\| \ \|_2 = | \ |$, the usual absolute value norm, discussed above, and $\| \ \|_1$ is arbitrary. There is nothing to prove if $v = 0$, since any positive constants a and b work in that case. Assume from now an that

$v \neq 0$. Then, dividing through by $|v|$ and using properties of the norm, we see that $a|v| \leq \|v\|_1 \leq b|v|$ is equivalent to $a \leq \|u\|_1 \leq b$ where $u = v/|v|$ has (usual) norm $|u| = 1$.

Step 3. Show that $v \to \|v\|_1$ is a continuous function with respect to $|\ |$. That is, given $v \in V$ and $\varepsilon > 0$, show there exists $\delta > 0$ such that if $w \in V$ and $|v - w| < \delta$, then

$$|\, \|v\|_1 - \|w\|_1 \,| < \varepsilon.$$

Step 4. Apply the *extreme value theorem*, a continuous function on a compact set (closed and bounded) achieves a minimum and a maximum value. In our case, the compact set is $\{u \in V : \|u\|_1 = 1\}$ and the minimum and maximum values are the desired constants a and b, respectively. $\square$

Definition. The *operator norm* on the vector space $M_n(F)$ of $n \times n$ matrices with coefficients in F is given by

$$\|A\| := \max_{|x| \leq 1} |Ax|.$$

for each $A \in M_n(F)$ where $|\ |$ is the usual norm on F.

Remarks.

1. For the identity matrix, we have $\|I_n\| = 1$.

2. The real number $\|A\|$ is the most that A scales any vector:

$$\|A\| = \max_{x \neq 0} A\left(\frac{x}{|x|}\right) = \max_{x \neq 0} \frac{|Ax|}{|x|}.$$

   Thus, $|Ax| \leq \|A\||x|$ for all $x \in F^n$. A detailed proof will be given below.

3. When trying to define a norm on $M_n(F)$, it might seem more natural to just think of an $n \times n$ matrix as an element of F^{n^2} and use the usual norm on F^{n^2}. However, the norm we have just described is easier to work with and, according to the proposition given above, it is equivalent to any other norm on $M_n(F)$.

Lemma 1. For all $A, B \in M_n(F)$ and $x \in F^n$,

1. $|Ax| \leq \|A\||x|$.

2. $\|AB\| \leq \|A\|\|B\|$.

3. $\|A^k\| \leq \|A\|^k$.

Proof. For part 1, first note that the inequality holds when $x = 0$. So suppose that $x \neq 0$, and let $u = \frac{x}{|x|}$. We have that $|u| = 1$, and hence,

$$\frac{|Ax|}{|x|} = \left| A\frac{x}{|x|} \right| = |Au| \leq \max_{|y|\leq 1} |Ay| = \|A\|.$$

Multiplying through by $|x|$ gives $|Ax| \leq \|A\||x|$, as desired.

For part 2, note that for all $x \in F^n$ with $|x| \leq 1$, we have from part 1,

$$|(AB)(x)| = |A(Bx)| \leq \|A\||Bx| \leq \|A\|\|B\||x| \leq \|A\|\|B\|.$$

Therefore,

$$\|AB\| := \max_{|x|\leq 1} |(AB)(x)| \leq \|A\|\|B\|.$$

Part 3 follows from part 2. □

Definition. Let $(v_k)_{k=0,1,\ldots}$ be a sequence in a normed vector space $(V, \| \, \|)$. We say

$$\lim_k v_k = v$$

for some vector $v \in V$ if for all $\varepsilon > 0$ there exists $N \in \mathbb{R}$ such that

$$\|v - v_k\| < \varepsilon$$

whenever $k \geq N$. A series $\sum_{k=0}^{\infty} v_k$ converges to v if its sequence of partial sums v_0, $v_0 + v_1$, $v_0 + v_1 + v_2, \ldots$ converges to v.

Theorem. For all $A \in M_n(F)$ and $t_0 > 0$, the function $\mathbb{R} \to M_n(F)$ given by

$$t \mapsto \sum_{k\geq 0} \frac{A^k t^k}{k!}$$

converges absolutely and uniformly for $t \in [-t_0, t_0]$.

Before proving this theorem, let's review the notions of absolute and uniform convergence of series of functions. First, a series $\sum_k v_k$ in a normed vector space $(V, \| \, \|)$ is *absolutely convergent* if $\sum_k \|v_k\|$ converges. If a series is absolutely convergent then every rearrangement of the series will converge.

Let V and W be normed vector spaces, and let $C \subseteq W$. (For instance, we could take $W = \mathbb{R}$ and $C = [-t_0, t_0]$.) For each $n \geq 0$, let $f_n \colon W \to V$ be a function. The sequence (f_n) *converges uniformly* to $f\colon W \to V$ on C if for all $\varepsilon > 0$, there exists an $N(\varepsilon) \in \mathbb{R}$ such that for all $x \in C$,

$$\|f(x) - f_n(x)\| < \varepsilon$$

whenever $n > N(\varepsilon)$. **Note:** the word "uniform" refers to the fact that $N(\varepsilon)$ is independent of x.

The notion of uniform convergence makes sense for a series $\sum_k f_k$ since a series is just a sequence of partial sums.

Week 3, Wednesday: Fundamental theorem for linear systems

From now on, page references are to our text. Recall that we will always be working over the field $F = \mathbb{R}$ or $\mathbb{C}$.

Definition. A sequence (v_k) in a normed vector space $(V, \|\ \|)$ is a *Cauchy sequence* if for all $\varepsilon > 0$ there exists $N \in \mathbb{R}$ such that for all $m, n > N$, we have

$$\|v_n - v_m\| < \varepsilon.$$

A theorem from analysis says that if V is finite-dimensional then it is *complete*: a sequence (v_k) converges if and only if it is a Cauchy sequence.

Lemma. (Weierstrass M-test) Let V and W be normed vector spaces with V finite-dimensional. For each $k \geq 0$, let $f_k \colon W \to V$ be a function. Let $C \subseteq W$, and suppose there exists a sequence $(M_k)_k$ of positive numbers such that

$$\|f_k(x)\| \leq M_k$$

for all $x \in C$ and for all k. Suppose further that $\sum_k M_k$ converges. Then $\sum_k f_k$ is absolutely and uniformly convergent on C.

Proof. A sequence in a normed space over F converges if and only if it's a Cauchy sequence. Let $\varepsilon > 0$. Since $\sum_k M_k$ converges, there exists $N \in \mathbb{R}$ such that for all $n \geq m > N$, we have

$$|\textstyle\sum_{k=0}^{n} M_k - \sum_{k=0}^{m} M_k| = |\sum_{k=m+1}^{n} M_k| < \varepsilon.$$

But then for $n \geq m > N$ is follows that for all $x \in C$

$$\|\textstyle\sum_{k=m+1}^{n} f_k(x)\| \leq \sum_{k=m+1}^{n} \|f_k(x)\| \leq \sum_{k=m+1}^{n} M_k < \varepsilon.$$

Thus $\sum_k f_k$ is uniformly Cauchy. $\square$

We are now ready to prove that it makes sense to exponentiate a matrix:

Theorem. For all $A \in M_n(F)$ and $t_0 > 0$, the function $\mathbb{R} \to M_n(F)$ given by

$$t \mapsto \sum_{k\geq 0} \frac{A^k t^k}{k!}$$

converges absolutely and uniformly for $t \in [-t_0, t_0]$.

Proof. Let $a := \|A\|$ and suppose that $|t| \leq t_0$. Then from Lemma 1 in the previous lecture,

$$\left\| \frac{A^k t^k}{k!} \right\| \leq \frac{\|A\|^k |t|^k}{k!} \leq \frac{\|A\|^k t_0^k}{k!} = \frac{a^k t_0^k}{k!} =: M_k.$$

It follows that

$$\sum_{k\geq 0} M_k = e^{at_0},$$

the usual exponential function. The result follows by the Weierstrass M-test. $\square$

Definition. Let $A \in M_n(F)$ and $t \in \mathbb{R}$. Then

$$e^{At} := \sum_{k\geq 0} \frac{A^k t^k}{k!}.$$

Note: The proof of the previous theorem shows that e^{At} is absolutely convergent and uniformly convergent on any closed interval for t. Further,

$$\|e^{At}\| \leq e^{\|A\||t|}.$$

To rigorously prove this last statement, note that

$$\left\| \sum_{k=0}^{n} \frac{A^k t^k}{k!} \right\| \leq \sum_{k=0}^{n} \left\| \frac{A^k t^k}{k!} \right\| = \sum_{k=0}^{n} \frac{\|A\|^k |t|^k}{k!}$$

The norm is a continuous function and hence commutes with limits, and limits preserve inequalities. It therefore follows that

$$\|e^{At}\| = \left\| \lim_{n\to\infty} \sum_{k=0}^{n} \frac{A^k t^k}{k!} \right\| = \lim_{n\to\infty} \left\| \sum_{k=0}^{n} \frac{A^k t^k}{k!} \right\| = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{\|A\|^k |t|^k}{k!} = e^{\|A\||t||}.$$

Proposition. (p. 13) Let $A, P \in M_n(F)$ with P invertible. Then

$$e^{P^{-1}AP} = P^{-1} e^A P.$$

Proof. Recall the trick from linear algebra:

$$\begin{aligned}(P^{-1}AP)^k &= (P^{-1}AP)(P^{-1}AP)(P^{-1}AP)\cdots(P^{-1}AP)\\ &= P^{-1}A(PP^{-1})A(PP^{-1})A(P\cdots P^{-1})AP\\ &= P^{-1}A^kP.\end{aligned}$$

Therefore,

$$\begin{aligned}e^{P^{-1}AP} &:= \sum_{k\geq 0}\frac{(P^{-1}AP)^k}{k!}\\ &= \sum_{k\geq 0}\left(P^{-1}\frac{A^k}{k!}P\right)\\ &= P^{-1}\left(\sum_{k\geq 0}\frac{A^k}{k!}\right)P\\ &= P^{-1}e^AP.\end{aligned}$$

The matrices P^{-1} and P can be pulled out of the sum since multiplication by these represent linear transformations, which are continuous, and the sum is a limit—limits commute with continuous functions (by definition of continuity). □

Proposition. (p. 13) Let $A, B \in M_n(F)$. If A and B commute, then $e^{(A+B)} = e^Ae^B$.

Proof.

$$\begin{aligned}e^{(A+B)} &= \sum_{n\geq 0}\frac{1}{n!}(A+B)^n\\ &= \sum_{n\geq 0}\frac{1}{n!}\left(\sum_{i+j=n}\frac{n!}{i!j!}A^iB^j\right)\\ &= \sum_{i\geq 0}\frac{1}{i!}A^i\left(\sum_{j\geq 0}\frac{1}{j!}B^j\right)\\ &= e^Ae^B.\end{aligned}$$

□

Corollary. (p. 13) If $A \in M_n(F)$, then

$$e^{-A} = \left(e^{A}\right)^{-1}.$$

Proof. Since A and $-A$ commute,

$$I_n = e^0 = e^{(A+(-A))} = e^A e^{-A}.$$

□

Example. The above proposition only holds, in general, if the matrices A and B commute. Consider,

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

It is easy to check that $AB \neq BA$.
Since $A^k = 0$ for $k > 1$,

$$e^A = I + A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and

$$e^B = \sum_{k \geq 0} \frac{1}{k!} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^k = \sum_{k \geq 0} \frac{1}{k!} \begin{pmatrix} 1^k & 0 \\ 0 & 2^k \end{pmatrix} = \sum_{k \geq 0} \begin{pmatrix} 1/k! & 0 \\ 0 & 2^k/k! \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & e^2 \end{pmatrix}.$$

Thus,

$$e^A e^B = \begin{pmatrix} e & e^2 \\ 0 & e^2 \end{pmatrix}.$$

On the other hand, you can check by induction that

$$(A+B)^k = \begin{pmatrix} 1 & 2^k - 1 \\ 0 & 2^k \end{pmatrix}.$$

Hence,

$$e^{A+B} = \sum_{k \geq 0} \frac{1}{k!} \begin{pmatrix} 1 & 2^k - 1 \\ 0 & 2^k \end{pmatrix} = \begin{pmatrix} e & e^2 - e \\ 0 & e^2 \end{pmatrix} \neq e^A e^B.$$

Week 3, Friday: Fundamental theorem for linear systems. Linear systems in $\mathbb{R}^2$

Lemma. (p. 17) Let $A \in M_n(F)$. Then

$$\frac{d}{dt}e^{At} = Ae^{At}.$$

Proof. For any constants t and h, we know At and Ah commute. Therefore,

$$\begin{aligned}\frac{d}{dt}e^{At} &= \lim_{h\to 0}\frac{e^{A(t+h)} - e^{At}}{h}\\ &= \lim_{h\to 0}\frac{e^{At}e^{Ah} - e^{At}}{h}\\ &= \lim_{h\to 0} e^{At}\frac{e^{Ah} - I_n}{h}.\end{aligned}$$

Multiplication by a matrix is a linear and, hence, continuous transformation, and by definition, continuous functions commute with limits. So, continuing from above,

$$\begin{aligned}\frac{d}{dt}e^{At} &= \lim_{h\to 0} e^{At}\frac{e^{Ah} - I_n}{h}\\ &= e^{At}\lim_{h\to 0}\frac{e^{Ah} - I_n}{h}.\end{aligned}$$

We now use the fact that e^{Ah} is absolutely an uniformly convergent for h restricted to a compact set, e.g., for $h \in [-1, 1]$. This means, roughly, the we can manipulate the infinite sum defining the exponential as if it were a polynomial:

$$\frac{d}{dt}e^{At} = e^{At}\lim_{h\to 0}\frac{e^{Ah} - I_n}{h}$$

$$
\begin{aligned}
&= e^{At} \lim_{h \to 0} \frac{1}{h} \left(Ah + \frac{A^2 h^2}{2!} + \frac{A^3 h^3}{3!} + \dots \right) \\
&= e^{At} A \\
&= A e^{At}.
\end{aligned}
$$

The final step follows since A commutes with itself. $\square$

Theorem. (The Fundamental Theorem for Linear Systems. (p. 17)) Let $A \in M_n(F)$, and let $x_0 \in F^n$. The initial value problem

$$
\begin{aligned}
x' &= Ax \\
x(0) &= x_0
\end{aligned}
$$

has the unique solution

$$x = e^{At} x_0.$$

Proof. Using the preceding lemma, if $x(t) := e^{At} x_0$, then

$$
\begin{aligned}
x'(t) &= \frac{d}{dt} x(t) \\
&= \frac{d}{dt} \left(e^{At} x_0 \right) \\
&= \left(\frac{d}{dt} e^{At} \right) x_0 \\
&= A e^{At} x_0 \\
&= Ax.
\end{aligned}
$$

Further, $x(0) = e^0 x_0 = x_0$. For uniqueness, suppose that $x(t)$ is any solution, and consider $y(t) := e^{-At} x(t)$. By the product rule,

$$
\begin{aligned}
y'(t) &= \left(e^{-At} \right)' x(t) + e^{-At} x'(t) \\
&= -A e^{-At} x(t) + e^{-At} \left(A x(t) \right) \\
&= e^{-At} \left(-A x(t) + A x(t) \right) \\
&= 0.
\end{aligned}
$$

Therefore $y(t)$ is constant. To determine the constant, let $t = 0$:

$$y(0) = e^0 x(0) = I_n x_0 = x_0.$$

Then,

$$y(t) = e^{-At}x(t) = x_0 \quad \Rightarrow \quad x(t) = e^{At}x_0.$$

$\square$

TWO-DIMENSIONAL LINEAR SYSTEMS

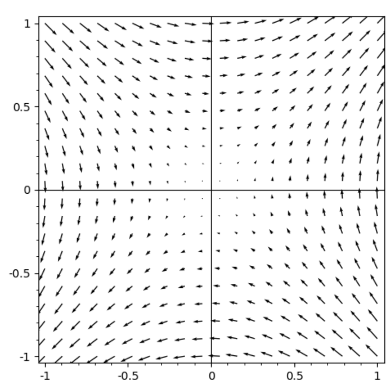
Example. Consider the (coupled) linear system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_1. \end{aligned}$$

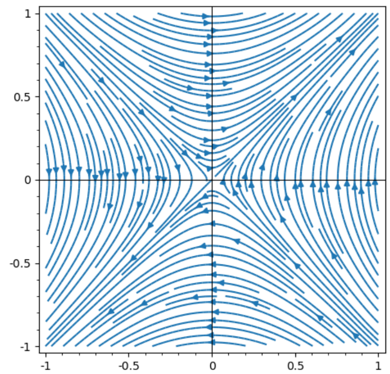
Given an initial condition (a, b), a solution will be a curve $x(t) = (x_1(t), x_2(t))$ in the plane, passing through (a, b) at time $t = 0$. The system itself tells us the velocity vector of any potential solution at every time:

$$x'(t) = (x_1'(t), x_2'(t)) = (x_2(t), x_1(t)).$$

So the system determines the vector field $F(x_1, x_2) = (x_2, x_1)$ on $\mathbb{R}^2$, pictured below:



Any solution curve must "follow the flow", i.e., its velocity vectors coincide with those already drawn above. Some possible solution curves are drawn below. You can see the paths of the curves but not their speeds:



We will now solve the system using the tools we have developed. First write the system as $x' = Ax$ with $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The solution is $x = e^{At}x_0$ where $x_0 = x(0)$. In order to exponentiate A, we diagonalize it. The characteristic polynomial of A is

$$\det(A - xI_2) = \det \begin{pmatrix} -x & 1 \\ 1 & -x \end{pmatrix} = x^2 - 1 = (x+1)(x-1).$$

So the eigenvalues are ± 1. It's easy to eyeball the corresponding eigenvectors: $(1, 1)$ and $(1, -1)$, respectively. So let

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Then

$$P^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

and $P^{-1}AP = \operatorname{diag}(1, -1) =: D$.
Therefore, $D = PAP^{-1}$, and

$$e^{At} = e^{PDP^{-1}t} = e^{P(Dt)P^{-1}} = Pe^{Dt}P^{-1}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} e^t + e^{-t} & e^t - e^{-t} \\ e^t - e^{-t} & e^t + e^{-t} \end{pmatrix}.$$

So, for example, the solution with initial condition $x(0) = (1, 0)$ is

$$x(t) = e^{At}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^t + e^{-t} & e^t - e^{-t} \\ e^t - e^{-t} & e^t + e^{-t} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^t + e^{-t} \\ e^t - e^{-t} \end{pmatrix}.$$

To see what is happening geometrically, note that

$$x' = Ax = PDP^{-1}x \quad \Rightarrow \quad P^{-1}x' = DP^{-1}x.$$

Letting $y := P^{-1}x$, we have $y' = P^{-1}x'$. So substituting gives

$$y' = Dy = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} y$$

an uncoupled system:

$$\begin{aligned} y_1' &= y_1 \\ y_2' &= -y_2 \end{aligned}$$

with solution $y_1 = ae^t$ and $y_2 = be^{-t}$. We then get the solution to our original equation by

$$x = Py.$$

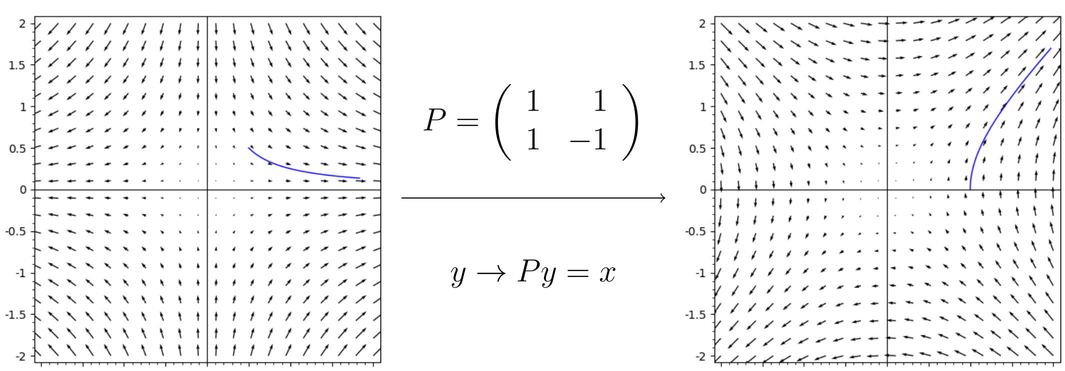
The initial condition $x(0) = (1, 0)$ in the x-coordinates transforms to the initial condition

$$y(0) = P^{-1}x(0) = P^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

in the y-coordinates, which implies $a = b = \frac{1}{2}$. So in the y-coordinates, our solution is

$$y(t) = \frac{1}{2}(e^t, e^{-t}).$$

The geometry is shown below:



Question. How is the magnitude and sign of the determinant of P expressed in the above image?

Week 4, Monday: Linear systems in $\mathbb{R}^2$

LINEAR SYSTEMS IN $\mathbb{R}^2$

Let $A \in M_2(\mathbb{R})$. The characteristic polynomial has real coefficients and degree 2. That means that if λ is a complex eigenvalue for A (with nonzero imaginary part), then so is its conjugate $\bar{\lambda}$. Otherwise, A either has two distinct real eigenvalues or one real eigenvalue with multiplicity 2. In order to exponentiate A, it would be nice to conjugate A (i.e., apply the mapping $A \to P^{-1}AP$ for some P) to a matrix that is close to being diagonal. We will discuss the Jordan form more carefully later, but for now it suffices to know that there exists an invertible real matrix P such that $P^{-1}AP$ has one of the three possible forms below:

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}, \qquad \begin{pmatrix} u & 1 \\ 0 & u \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

where $u, v, a, b \in \mathbb{R}$. The first case occurs when A has eigenvalues u and v (including the case where $u = v$ occurs with multiplicity 2) and A is diagonalizable. The second case occurs when A has the real eigenvalue u with multiplicity 2 but the corresponding eigenspace only has dimension 1. The last case occurs when A has a pair of complex eigenvalues $\lambda = a + bi$ and $\lambda = a - bi$. (If we were working over $\mathbb{C}$, then in this last case A could be conjugated to the diagonal matrix $\operatorname{diag}(\lambda, \bar{\lambda})$, as we will discuss below.)

To solve two-dimensional linear systems, we need to exponentiate matrices with these forms. The first is easy:

$$\exp \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} e^u & 0 \\ 0 & e^v \end{pmatrix}.$$

For the second, let's exponentiate a slightly more general matrix:

$$B := \begin{pmatrix} u & v \\ 0 & u \end{pmatrix}.$$

Let

$$C = \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix},$$

and note that (i) $B = uI + C$, (ii) $C^k = 0$ for $k > 1$, and (iii) uI and C commute. It follows that

$$\begin{aligned}
e^B = e^{uI+C} = e^{uI}e^C = \begin{pmatrix} e^u & 0 \\ 0 & e^u \end{pmatrix} e^C = e^u I e^C = e^u e^C \\
&= e^u \left(I + C + \frac{1}{2}C^2 + \frac{1}{3!}C^3 + \dots \right) \\
&= e^u \left(I + C \right) \\
&= \begin{pmatrix} e^u & ve^u \\ 0 & e^u \end{pmatrix}.
\end{aligned}$$

Now consider the last case, in which

$$J = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Letting

$$Q = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$$

we have

$$\begin{aligned}
Q^{-1}JQ &= \frac{1}{2i} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \\
&= \frac{1}{2i} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} \begin{pmatrix} ai - b & -ai - b \\ a + bi & a - bi \end{pmatrix} \\
&= \frac{1}{2i} \begin{pmatrix} 2ai - 2b & 0 \\ 0 & 2ai + 2b \end{pmatrix} \\
&= \begin{pmatrix} a + bi & 0 \\ 0 & a - bi \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}.
\end{aligned}$$

Therefore, using the fact that

$$e^{\lambda t} = e^{at+bti} = e^{at}(\cos(bt) + i\sin(bt)) \quad \text{and} \quad e^{\bar{\lambda} t} = e^{at-bti} = e^{at}(\cos(bt) - i\sin(bt)),$$

we have

$$e^{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} t} = Qe^{\mathrm{diag}(\lambda,\bar{\lambda})t}Q^{-1}$$

$$= \frac{1}{2i} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\bar{\lambda} t} \end{pmatrix} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}$$

$$= \frac{1}{2i} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{\lambda t} & ie^{\lambda t} \\ -e^{\bar{\lambda} t} & ie^{\bar{\lambda} t} \end{pmatrix}$$

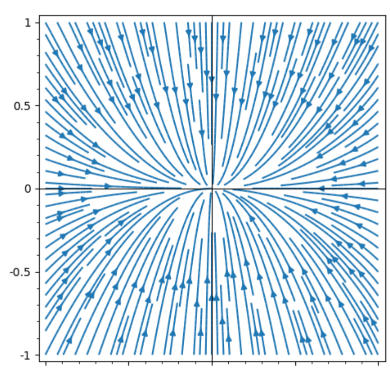
$$= \frac{1}{2i} \begin{pmatrix} ie^{\lambda t} + ie^{\bar{\lambda} t} & -e^{\lambda t} + e^{\bar{\lambda} t} \\ e^{\lambda t} - e^{\bar{\lambda} t} & ie^{\lambda t} + ie^{\bar{\lambda} t} \end{pmatrix}$$

$$= e^{at} \begin{pmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{pmatrix}.$$

Let's look at the corresponding systems of differential equations and their solutions with initial condition x_0:
If $J = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ then the solution is

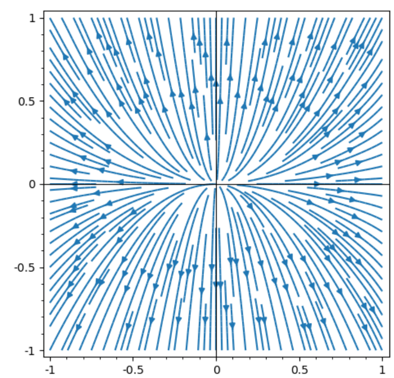
$$x(t) = \begin{pmatrix} e^{ut} & 0 \\ 0 & e^{vt} \end{pmatrix} x_0.$$

If both u and v are negative, the origin is a *stable node* ($u = -1, v = -2$ displayed):



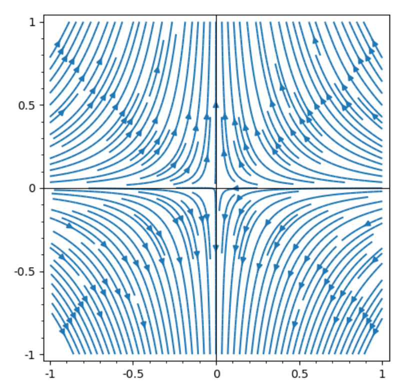
.

If u and v are both positive, the origin is an *unstable node* ($u = 1, v = 2$ displayed):



.

If one of u and v is negative and the other is positive, the origin is a *saddle point* ($u = -1, v = 2$ displayed):



.

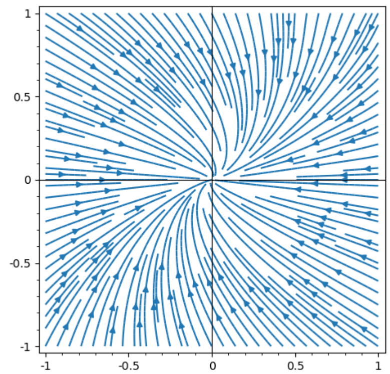
If $J = \begin{pmatrix} u & 1 \\ 0 & u \end{pmatrix}$ then

$$e^{Jt} = \exp \begin{pmatrix} u & t \\ 0 & u \end{pmatrix} = \begin{pmatrix} e^u & te^u \\ 0 & e^u \end{pmatrix}$$

and the solution is

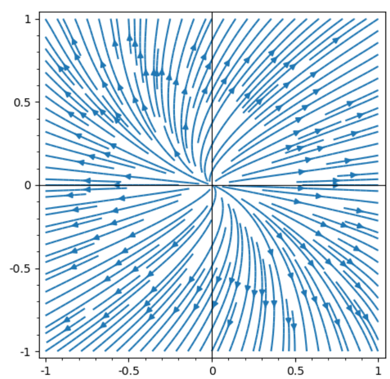
$$x(t) = \begin{pmatrix} e^{ut} & te^{ut} \\ 0 & e^{ut} \end{pmatrix} x_0.$$

If $u < 0$, the origin is a *stable node* ($u = -2$ displayed):



.

and if it is positive, then the origin is an *unstable node* ($u = 2$ displayed):

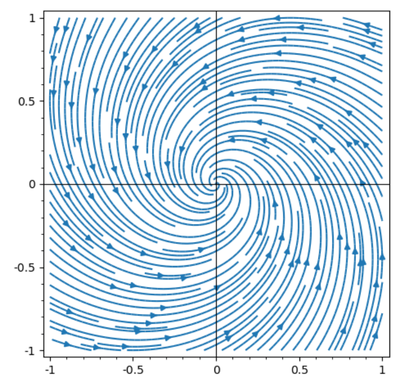


.

If $J = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ then the solution is

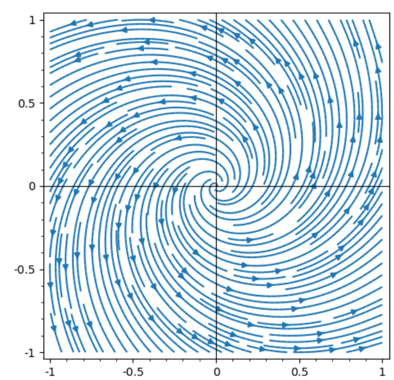
$$x(t) = e^{at} \begin{pmatrix} \cos(b) & -\sin(b) \\ \sin(b) & \cos(b) \end{pmatrix} x_0.$$

If $a < 0$, then each solution spirals into the origin and we say the origin is a *stable focus* ($a = -1, b = 2$ displayed):



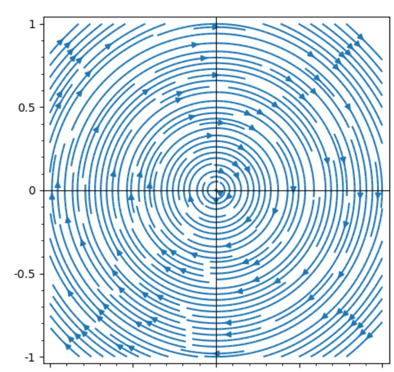
.

If $a > 0$, then each solution spirals away from the origin, and we say the origin is an *unstable focus* ($a = 1, b = 2$ displayed):



.

If $a = 0$, each solution goes in a circle about the origin, and we say that the system has a *center* at the origin ($a = 0, b = -2$ displayed):



.

In any of these cases, if $b > 0$ the motion is counterclockwise, and if $b < 0$, the motion is clockwise.

We've discussed all cases in which both eigenvalues are nonzero. If either of the eigenvalues is zero, i.e., if $\det(A) = 0$, then the origin is a *degenerate equilibrium point*. See our text for pictures of these systems.

Lemma. Let $A \in M_n(F)$ with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then

1. $\mathrm{trace}(A) := \sum_{i=1}^n A_{ii} = \sum_{i=1}^n \lambda_i$ and $\det(A) = \prod_{i=1}^n \lambda_i$.
2. Consider the characteristic polynomial of A:
$$p(x) = \det(A - xI_n).$$
Then the coefficient of x^{n-1} in $p(x)$ is $(-1)^{n-1}\mathrm{trace}(A)$ and the constant term of $p(x)$ is $\det(A)$.[1]

Proof. Recall that for all $C, D \in M_n(F)$, we have

$$\mathrm{trace}(CD) = \mathrm{trace}(DC)$$

and

$$\det(CD) = \det(C)\det(D) = \det(D)\det(C) = \det(DC).$$

Therefore, for all invertible $P \in M_n(F)$,

$$\mathrm{trace}(P^{-1}AP) = \mathrm{trace}(A) \quad \text{and} \quad \det(P^{-1}AP) = \det(A).$$

Further, the characteristic polynomial is not affected by conjugation:

$$\det(P^{-1}AP - xI_n) = \det(P^{-1}(A - xI_n)P) = \det(P^{-1})\det(A - xI_n)\det(P) = \det(A - xI).$$

Therefore, we may assume that A is in Jordan form—an upper triangular matrix. Considering $p(x) = \det(A - xI)$, we see the diagonal entries are the eigenvalues, $\lambda_1 \ldots, \lambda_n$. Part 1 follows. Next, consider the characteristic polynomial

$$\det(A - xI_x) = p(x) = (\lambda_1 - x)\cdots(\lambda_n - x).$$

Expanding the right-hand side, we see that the coefficient of x^{n-1} is $\mathrm{trace}(A)$. Setting $x = 0$ in the above equation then completes the proof of part 2. $\square$

Let's now go back to the case $n = 2$. Let $\tau := \mathrm{trace}(A)$ and $\delta := \det(A)$. Up to conjugation, there are three possibilities:

[1] The characteristic polynomials is sometimes defined to be $p(x) = \det(xI_n - A)$. In that case, the coefficient of x^{n-1} is $-\mathrm{trace}(A)$. The constant term is again $\det(A)$.

$$\begin{array}{ccc} \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} & \begin{pmatrix} u & 1 \\ 0 & u \end{pmatrix} & \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \\ \tau = u + v & \tau = 2u & \tau = 2a \\ \delta = uv & \delta = u^2 & \delta = a^2 + b^2 \end{array}$$

The characteristic polynomial is

$$p(x) = x^2 - \tau\, x + \delta.$$

So the eigenvalues are

$$\frac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2}. \tag{10.1}$$

Theorem. (p. 25)

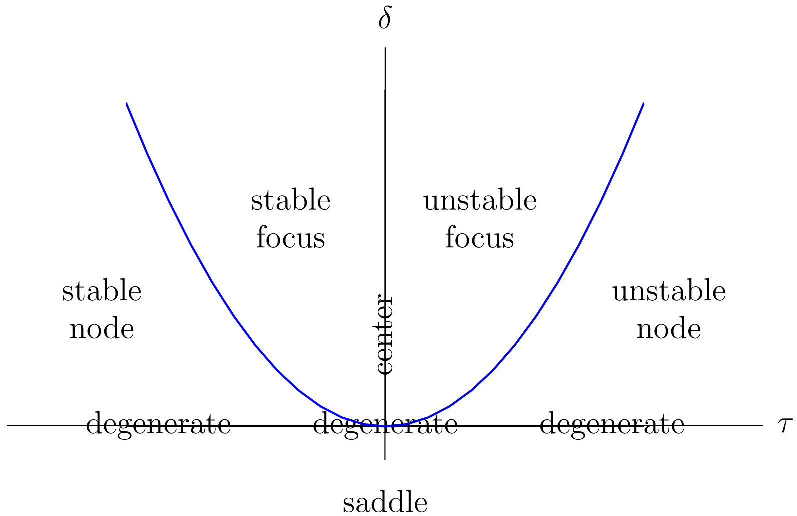
1. If $\delta < 0$, then the origin is a saddle point.

2. If $\delta > 0$ and $\tau^2 - 4\delta \geq 0$, then the origin is a stable node if $\tau < 0$ and an unstable node if $\tau > 0$. (Note that in this case, the conditions $\delta > 0$ and $\tau^2 - 4\delta \geq 0$ imply $\tau \neq 0$.)

3. If $\delta > 0$ and $\tau^2 - 4\delta < 0$, then the origin is a stable focus if $\tau < 0$, an unstable focus if $\tau > 0$, or a center if $\tau = 0$ (in which case $\tau - 4\delta < 0$ is automatic).

Proof. If $\delta < 0$, then equation 10.1 shows that one eigenvalue is positive and the other is negative. Hence, the origin is a saddle point. That proves the first part. The others follow similarly. □

Calling a stable node or focus a *sink* and calling an unstable node or focus a *source*, we get the following diagram:



δ
τ
stable focus
unstable focus
stable node
unstable node
center
degenerate
degenerate
degenerate
saddle

Week 4, Wednesday: Jordan form

Review of diagonalization. For a diagonal matrix $D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, we have

$$De_i = \lambda_i e_i$$

for each standard basis vector e_i. If $A \in M_n(F)$ is not diagonal, we look for linearly independent vectors that behave like the e_i above:

$$Av_i = \lambda_i v_i.$$

If we can find n of these vectors, then changing to the basis $\{v_1, \dots, v_n\}$, these v_i are transformed to the standard basis vectors in the new coordinates, and A is diagonalized.

Therefore, we look for vectors $v \neq 0$ such that

$$Av = \lambda v$$

for some $\lambda \in F$. We have

$$Av = \lambda v \quad \Leftrightarrow \quad (A - \lambda I_n)v = 0 \quad \Leftrightarrow \quad v \in \ker(A - \lambda I_n).$$

The kernel is nonzero if and only if $\det(A - \lambda I_n) = 0$. So to find suitable λ, the *eigenvalues*, we consider the *characteristic polynomial*

$$p(x) = \det(A - xI_n) = \prod_{j=1}^{n} (\lambda_j - x) = \prod_{j=1}^{\ell} (\mu_j - x)^{k_j}.$$

In the expression on the far right, repeated eigenvalues are grouped together (so each μ_j is equal to some λ_t). The *algebraic multiplicity* of the eigenvalue μ_j is k_j. The eigenvectors corresponding to μ_j form a subspace of F^n called the *eigenspace* for μ_j:

$$E_{\mu_j} := \ker(A - \mu_j I_n).$$

The dimension of E_{μ_j} is the *geometric multiplicity* of μ_j. We always have that the geometric multiplicity is at most the algebraic multiplicity:

$$\dim E_{\mu_j} \le k_j.$$

The matrix A is diagonalizable if and only if there is a basis consisting of eigenvectors, and that happens exactly when the geometric multiplicity of each eigenvalue equals it algebraic multiplicity. If that is not the case, we can still choose bases for each eigenspace, but we are then left with the task of completing this set to a full basis for F^n. By choosing correctly, we can assure that A has a nice form.

Jordan Form

Let $\lambda \in F$. A $k \times k$ *Jordan block* for λ is a $k \times k$ matrix with λ appearing along the diagonal and 1s appearing on the superdiagonal:

$$J_k(\lambda) := \begin{pmatrix} \lambda & 1 & & & & \\ & \lambda & 1 & & \text{\Large 0} & \\ & & \lambda & 1 & & \\ & & & \ddots & & \\ & \text{\Large 0} & & & & 1 \\ & & & & & \lambda \end{pmatrix}.$$

For example,

$$J_4(2) = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

A *Jordan matrix* is a square block-diagonal matrix with Jordan matrices along the diagonal:

$$J := \begin{pmatrix} J_{k_1}(\lambda_1) & & & & \\ & J_{k_2}(\lambda_2) & & \text{\Large 0} & \\ & & J_{k_3}(\lambda_3) & & \\ & \text{\Large 0} & & \ddots & \\ & & & & J_{k_\ell}(\lambda_\ell). \end{pmatrix}.$$

For example, the following is a Jordan matrix:

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2+3i \end{pmatrix}$$

with Jordan blocks $J_1(2), J_1(2), J_3(4), J_2(i)$ and $J_1(2+3i)$.

A diagonal matrix is a Jordan matrix whose Jordan blocks are all 1×1.

Theorem. Let $A \in M_n(\mathbb{C})$. Then there exists an invertible matrix $P \in M_n(\mathbb{C})$ such that $P^{-1}AP = J$ where J is a Jordan matrix. The matrix J is called the *Jordan form for* A. It is unique up to a permutation of the Jordan blocks. The diagonal entries of J are exactly the eigenvalues of A repeated according to their algebraic multiplicities (the number of times the eigenvalue appears in a factorization of the characteristic polynomial of A over $\mathbb{C}$). The number of blocks having a particular eigenvalue λ along the diagonal is the geometric multiplicity of λ (i.e., $\dim(A - \lambda I_n)$).

Example. A matrix is diagonalizable if and only if each of its Jordan blocks is 1×1. For example, we know

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is not diagonalizable since it is already in Jordan form and it's not diagonal. The matrix A has one eigenvalue, 1, of multiplicity 2, but the dimension of the eigenspace for 1 is 1-dimensional:

$$\ker(A - 1 \cdot I_2) = \ker \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \{(x, 0) : x \in F\},$$

which has basis $\{(1, 0)\}$. As claimed the number of Jordan blocks for 1 is the geometric multiplicity of 1.

Jordan form over the reals. Now suppose that $A \in M_n(\mathbb{R})$. Then it turns out that we can conjugate A via a real matrix it to a real matrix that is almost as nice as the Jordan form over $\mathbb{C}$. Since A is defined over the reals, its nonreal eigenvalues appear in conjugate pairs, and it turns out that each $k \times k$ Jordan block for $\lambda = a + bi$ has a corresponding $k \times k$ Jordan block for $\bar{\lambda} = a - bi$ of the same dimension. We can combine these blocks and change basis to get a corresponding $2k \times 2k$ block matrix with 2×2 blocks of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

along the diagonal, and the 2×2 identity matrix I_2 appearing along the super diagonal. For instance, the Jordan matrix

$$\begin{pmatrix} \lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\lambda} & 1 & 0 \\ 0 & 0 & 0 & 0 & \bar{\lambda} & 1 \\ 0 & 0 & 0 & 0 & 0 & \bar{\lambda} \end{pmatrix}$$

where $\lambda = a + bi$ can be conjugated to the form

$$\begin{pmatrix} a & -b & 1 & 0 & 0 & 0 \\ b & a & 0 & 1 & 0 & 0 \\ 0 & 0 & a & -b & 1 & 0 \\ 0 & 0 & b & a & 0 & 1 \\ 0 & 0 & 0 & 0 & a & -b \\ 0 & 0 & 0 & 0 & b & a \end{pmatrix}$$

If $A \in M_n(\mathbb{R})$, there exists an invertible $P \in M_n(\mathbb{R})$ such that $P^{-1}AP = J$ where J consists of Jordan blocks—the usual ones for real eigenvalues, and these modified block matrices for conjugate pairs of complex eigenvalues. The form is unique up to permutation of the blocks and swaps

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \longleftrightarrow \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

We will call it the *real Jordan form* for A. Here is a typical real Jordan form for a real matrix:

$$\begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 3 \end{pmatrix}$$

There are two Jordan blocks for 4: one is 1×1 and one is 3×3. The other eigenvalues for this matrix are $3+2i$ and $3-2i$, each of which appears with multiplicity 3. Notice there are two real Jordan blocks for the pair $3 \pm 2i$, one is 2×2 and the other is 4×4.

Week 4, Friday: Exponentiating Jordan matrices. Algorithm for computing Jordan form

EXPONENTIATION OF JORDAN MATRIX

To solve the linear system $x' = Ax$, we need to compute e^{At}. If $P^{-1}AP = J$ where J is the Jordan form of A, then $e^{At} = Pe^{Jt}P^{-1}$. Then, to exponentiate J, we must exponentiate each of its blocks. If

$$J := \begin{pmatrix} J_{k_1}(\lambda_1) & & & & \\ & J_{k_2}(\lambda_2) & & \mathbf{0} & \\ & & J_{k_3}(\lambda_3) & & \\ & \mathbf{0} & & \ddots & \\ & & & & J_{k_\ell}(\lambda_\ell) \end{pmatrix},$$

then

$$e^{Jt} := \begin{pmatrix} e^{J_{k_1}(\lambda_1)t} & & & & \\ & e^{J_{k_2}(\lambda_2)t} & & \mathbf{0} & \\ & & e^{J_{k_3}(\lambda_3)t} & & \\ & \mathbf{0} & & \ddots & \\ & & & & e^{J_{k_\ell}(\lambda_\ell t)} \end{pmatrix}.$$

Thus, we are reduced to exponentiating Jordan blocks, which we talk about here, starting with an example. Let $\lambda \in F$ and consider the Jordan block

$$J_4(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \lambda I_4 + N_4$$

where

$$N_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since λI_4 and N_4 commute,

$$e^{J_4(\lambda)t} = e^{(\lambda I_4 + N_4)t} = e^{\lambda t I_4} e^{tN_4}.$$

As usual,

$$e^{\lambda t I_4} = \begin{pmatrix} e^{\lambda t} & 0 & 0 & 0 \\ 0 & e^{\lambda t} & 0 & 0 \\ 0 & 0 & e^{\lambda t} & 0 \\ 0 & 0 & 0 & e^{\lambda t} \end{pmatrix} = e^{\lambda t} I_4.$$

So we are left with computing e^{tN_4}:

$$e^{N_4 t} = I_4 + tN_4 + \frac{t^2}{2!}N_4^2 + \frac{t^3}{3!}N_4^3 + \frac{t^4}{4!}N_4^4 + \frac{t^5}{5!}N_4^5 + \cdots$$

Consider the powers of N_4:

$$N_4^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$N_4^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$N_4^4 = 0.$$

All higher powers of N_4 are 0. Notice how as we take powers, the diagonal of 1s climbs up to the right along successively higher diagonals.

Returning to the calculation,

$$e^{J_4(\lambda t)} = e^{\lambda t}\left(I_4 + tN_4 + \frac{t^2}{2!}N_4^2 + \frac{t^3}{3!}N_4^3\right)$$

$$= e^{\lambda t}\begin{pmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} \\ 0 & 1 & t & \frac{t^2}{2!} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!}e^{\lambda t} & \frac{t^3}{3!}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!}e^{\lambda t} \\ 0 & 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & 0 & e^{\lambda t} \end{pmatrix}.$$

For instance, the solution to $x' = J_4(\lambda)x$ with initial condition $x_0 = (4, 3, 2, 1)$ is

$$\begin{aligned} x(t) &= e^{J_4(\lambda)t}x_0 \\ &= e^{\lambda t}\begin{pmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} \\ 0 & 1 & t & \frac{t^2}{2!} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} \\ &= e^{\lambda t}\begin{pmatrix} 4 + 3t + 2\frac{t^2}{2!} + \frac{t^3}{3!} \\ 3 + 2t + \frac{t^2}{3!} \\ 2 + t \\ 1 \end{pmatrix}, \end{aligned}$$

or

$$x(t) = e^{\lambda t}\left(4 + 3t + 2\frac{t^2}{2!} + \frac{t^3}{3!},\ 3 + 2t + \frac{t^2}{3!},\ 2 + t,\ 1\right).$$

Now consider a general Jordan block:

$$J_k(\lambda) = \lambda I_k + N_k$$

where N_k is the matrix with 1s along the superdiagonal. As before, taking powers of N_k causes the diagonal of 1 to march up to the right, and we get $N_k^k = 0$. A matrix N such that $N^k = 0$ is called *nilpotent*. The minimum k such that $N^k = 0$ is the *degree* of nilpotency. Thus, N_k is nilpotent of degree k. We have

$$\begin{aligned} e^{J_k(\lambda)t} &= e^{(\lambda I_k + N_k)t} = e^{\lambda t I_k}e^{N_k t} \\ &= e^{\lambda t}\left(I_k + tN_k + \frac{t^2}{2}N_k^2 + \frac{t^3}{3!}N_k^3 + \cdots + \frac{t^{k-1}}{(k-1)!}N_k^{k-1}\right) \end{aligned}$$

$$= e^{\lambda t}\begin{pmatrix} 1 & t & \frac{t^2}{2!} & \dots & \dots & \frac{t^{k-1}}{(k-1)!} \\ 0 & 1 & t & \dots & \dots & \frac{t^{k-2}}{(k-2)!} \\ 0 & 0 & 1 & \dots & \dots & \frac{t^{k-3}}{(k-3)!} \\ & \ddots & & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 & 1 & t \\ 0 & \dots & \dots & \dots & 0 & 1 \end{pmatrix}.$$

Note. If the real part of λ is negative, notice how

$$\lim_{t\to\infty} e^{J_k(\lambda)t} = 0.$$

Working exclusively over the reals, we will need to exponentiate Jordan blocks corresponding to pairs of conjugate eigenvalues. Let

$$M := \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

and consider a real Jordan block for $\lambda = a + bi$ with $b \neq 0$:

$$J := \begin{pmatrix} M & I_2 & 0 & \dots & \dots & 0 \\ 0 & M & I_2 & \dots & \dots & 0 \\ 0 & 0 & M & \dots & \dots & 0 \\ & \ddots & & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 & M & I_2 \\ 0 & \dots & \dots & \dots & 0 & M \end{pmatrix}.$$

To exponentiate, let

$$R := \begin{pmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{pmatrix}.$$

So

$$e^{Mt} = e^{at}R.$$

By an argument that is essentially the same as just given above, we get the matrix of 2×2 blocks

$$e^{Jt} = e^{at} \begin{pmatrix} R & tR & \frac{t^2}{2!}R & \dots & \dots & \frac{t^{k-1}}{(k-1)!}R \\ 0 & R & tR & \dots & \dots & \frac{t^{k-2}}{(k-2)!}R \\ 0 & 0 & R & \dots & \dots & \frac{t^{k-3}}{(k-3)!}R \\ & \ddots & & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 & R & tR \\ 0 & \dots & \dots & \dots & 0 & R \end{pmatrix}.$$

Again, notice that if $\mathrm{Re}(\lambda) = a < 0$, then

$$\lim_{t\to\infty} e^{Jt} = 0.$$

Algorithm for computing the Jordan form. Our book has a careful discussion of an algorithm for computing the Jordan form of a matrix A. We will not go into the details (unless there is demand for it!). Here, we'll give over a couple of points, though. To start the algorithm, compute the eigenvalues of the matrix by finding the zeros of the characteristic polynomial. We would like to know the number of Jordan blocks for each eigenvalue and their sizes. The key to this is as follows: Let λ be an eigenvalue, and consider the sequence of integers

$$\delta_\ell := \delta_\ell(\lambda) := \dim\ker(A - \lambda I)^\ell$$

for $\ell = 0, 1, 2, \dots$. **These δ_ℓ are invariant with respect to conjugation, so we might as well imagine that A is in Jordan form already and work block-by-block.** For a Jordan block $J_k(\mu)$ with $\mu \neq \lambda$,

$$\ker(J_k(\mu) - \lambda I)^\ell = 0$$

for all ℓ since each diagonal entry of each power is nonzero. So the $\delta_\ell(\lambda)$ for any block like this are all 0. Now consider each Jordan block of the form $J_k(\lambda)$. We have

$$\ker(J_k(\lambda) - \lambda I)^\ell = \ker N_k^\ell$$

where N_k is the nilpotent matrix from earlier. Thinking about the form of N_k^ℓ is it easy to see that the δ_ℓ sequence for blocks like these is

$$\delta_\ell = \begin{cases} \ell & \text{for } 0 \leq \ell \leq k, \\ k & \text{for } \ell > k. \end{cases}$$

k	$(A-\lambda I)^k$	**basis for kernel**	**dimension**
1	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	e_1	1
2	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	e_1, e_2	2
3	$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	e_1, e_2, e_3	3
4	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	e_1, e_2, e_3, e_4	4.

Figure 12.1: The case where $A = J_4(\lambda)$.

See Figure 12.1 for the case where $k = 4$.

The $\delta_\ell(\lambda)$-sequence for A is the sum of the $\delta_\ell(\lambda)$-sequences for each of its Jordan blocks. For instance, $\delta_1(\lambda)$ for A is the number of its Jordan blocks for λ—we've just seen that each of these contributes its $\delta_1 = 1$ to the count. With just a little more thought (see our text), letting ν_k be the number of $k \times k$ Jordan blocks for λ for the $n \times n$ matrix A, we get

$$\nu_k = \begin{cases} 2\delta_1 - \delta_2 & \text{for } k = 1, \\ 2\delta_k - \delta_{k+1} - \delta_{k-1} & \text{for } 1 < k < n, \\ \delta_n - \delta_{n-1} & \text{for } k = n. \end{cases}$$

The point is that the numbers of Jordan blocks of each size for each eigenvalue are determined by the δ-sequences, i.e, by the sequence of dimensions of the kernels, $\ker(A - \lambda I)^\ell$.

To actually conjugate A to Jordan form, for each eigenvalue λ, we consider the tower of subspaces

$$\ker(A-\lambda I) \subseteq \ker(A-\lambda I)^2 \subseteq \ker(A-\lambda I)^3 \subseteq \dots$$

Starting at the leftmost kernel in this tower of subsets, we could successively build bases for these kernels, adding vectors as we move to the right, as we could see earlier in the case where $A = J_4(\lambda)$. Appropriately chosen, these vectors are called *generalized eigenvectors*. We use them as columns of a matrix P so that $P^{-1}AP$ is the Jordan form for A

Let's consider the case where

$$A = J_4(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}.$$

Notice that we have

$$Ae_1 = \lambda e_1,\ Ae_2 = e_1 + \lambda e_2,\ Ae_3 = e_2 + \lambda e_3,\ Ae_4 = e_3 + \lambda e_4.$$

Therefore,

$$\begin{aligned}(A-\lambda I)e_1 &= 0 \\ (A-\lambda I)e_2 &= e_1 \\ (A-\lambda I)e_3 &= e_2 \\ (A-\lambda I)e_4 &= e_3,\end{aligned}$$

and $(A-\lambda I)^{i+1}e_i = 0$ for $i = 2, 3, 4$. So if A is not in Jordan form already, we will look for vectors $v_1, \dots, v_4$ that behave like the e_i, above. We need to solve $(A-\lambda I)v_i = v_{i-1}$ starting with v_1 an eigenvector with eigenvalue λ. These v_i will be columns in the matrix P.

Week 5, Monday: Stability theory. Linear systems in $\mathbb{R}^3$. Nonhomogeneous equations

Stability

Let $A \in M_n(\mathbb{C})$. For each eigenvalue $\lambda \in \mathbb{C}$, the *generalized eigenspace* for λ is

$$V_\lambda = \{v \in \mathbb{C}^n : (A - \lambda I)^k v = 0 \text{ for some } k > 0\}.$$

We can choose bases $\mathcal{B}_\lambda$ for the generalized eigenspaces resulting in a basis $\mathcal{B} = \cup_\lambda \mathcal{B}_\lambda$ for $\mathbb{C}^n$ with respect to which the matrix A attains its Jordan form. Define the *stable*, *center*, and *unstable* spaces for A respectively by

$$E^s = \text{Span } \cup_{\lambda:\text{Re}(\lambda)<0} \mathcal{B}_\lambda$$

$$E^c = \text{Span } \cup_{\lambda:\text{Re}(\lambda)=0} \mathcal{B}_\lambda$$

$$E^u = \text{Span } \cup_{\lambda:\text{Re}(\lambda)>0} \mathcal{B}_\lambda.$$

Since $\mathcal{B}$ is a basis, we can write

$$\mathbb{C}^n = E^s \oplus E^c \oplus E^u,$$

i.e., every $v \in \mathbb{C}^n$ can be written uniquely as $v = v_s + v_c + v_u$ where $v_s \in E^s$, $v_c \in E^c$, and $v_u \in E^u$.

If A is a real matrix and we are working over the real numbers, then define the (real) *stable*, *center*, and *unstable* spaces for A by intersecting each of E^s, E^c, and E^u with $\mathbb{R}^n$. Not that if A is real, is nonreal eigenvalues will occur in conjugate pairs $a \pm bi$, and the conjugates have the same real part. We can also adjust the basis $\mathcal{B}$ so that with respect to $\mathcal{B}$, the matrix A has its real Jordan form.

If $L : F^n \to F^n$ is a linear function and $W \subseteq F^n$, we say that W is *invariant under* L if $L(W) \subseteq W$. If M is the matrix representing L, we similarly say that W is invariant under M if $Mw \in W$ for all $w \in W$.

Proposition. Each generalized eigenspace, the stable, center, and unstable spaces are invariant under A and under e^{At} for all $t \in \mathbb{R}$.

Proof. Staring at the Jordan form for A and its exponential makes this result obvious, but we will give a formal proof. First consider the action of A. Fix an eigenvalue λ for A and consider the corresponding generalized eigenspace V_λ. Let $v \in V_\lambda$. To show that $Av \in V_\lambda$, we first let $w = (A - \lambda I)v$. We claim that $w \in V_\lambda$. To see this, take $k > 0$ such that $(A - \lambda I)^k v = 0$. Then $(A - \lambda I)^{k-1} w = 0$ (in the special case where $k = 1$, we have $(A - \lambda I)^0 w = (A - \lambda I)v = w = 0 \in V_\lambda$). Since $v, w \in V_\lambda$ and V_λ is a subspace,

$$Av = \lambda v + w \in V_\lambda.$$

This shows that V_λ is invariant under A. Now since each of the stable, center, and unstable spaces is formed by taking the linear span of bases for certain generalized eigenspaces, it follows that each of these is invariant under A. It follows that they are invariant under e^{At} by homework. $\square$

Thus, let $x(t)$ be the solution to the initial value problem $x' = Ax$, $x(0) = x_0$, i.e., let $x(t) = e^{At}x_0$. It follows that if $x_0 \in E^s$, then $x(t) \in E^s$ for all t. The solution never leaves the stable space. Similarly, a solution starting in the center or the unstable space never leaves that space. Further, from the Jordan form, one sees that

$$\begin{aligned} x_0 \in E^s \setminus \{0\} &\Longrightarrow \lim_{t\to\infty} x(t) = 0 \quad \text{and} \quad \lim_{t\to-\infty} |x(t)| = \infty \\ x_0 \in E^u \setminus \{0\} &\Longrightarrow \lim_{t\to\infty} |x(t)| = \infty \quad \text{and} \quad \lim_{t\to-\infty} x(t) = 0. \end{aligned}$$

In particular, if all eigenvalues of A have negative real part, then all solutions, no matter what the initial condition, are drawn into the origin. If all eigenvalues have positive real part, all solutions with non-zero initial condition will eventually leave any fixed compact set.

LINEAR SYSTEMS IN $\mathbb{R}^3$

Linear systems in $\mathbb{R}^3$. Let $A \in M_3(\mathbb{R})$. Then A either has three real eigenvalues (counting multiplicities) or it has a single real eigenvalue and pair of conjugate nonreal eigenvalues. Therefore, the possibilities for the Jordan form and for the solutions to $x' = Ax$ up to a linear change of coordinates are:

I. $u, v, w \in \mathbb{R}$:

$$J = \begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & w \end{pmatrix} \qquad x(t) = e^{Jt}x_0 = \begin{pmatrix} e^{ut} & 0 & 0 \\ 0 & e^{vt} & 0 \\ 0 & 0 & e^{wt} \end{pmatrix} x_0.$$

The behavior of the various trajectories will depend on the signs of u, v, w, with saddle-like behavior if they don't all have the same sign.

II. $u, v \in \mathbb{R}$:

$$J = \begin{pmatrix} u & 1 & 0 \\ 0 & u & 0 \\ 0 & 0 & v \end{pmatrix} \qquad x(t) = e^{Jt}x_0 = \begin{pmatrix} e^{ut} & te^{ut} & 0 \\ 0 & e^{ut} & 0 \\ 0 & 0 & e^{vt} \end{pmatrix} x_0.$$

III. $u \in \mathbb{R}$:

$$J = \begin{pmatrix} u & 1 & 0 \\ 0 & u & 1 \\ 0 & 0 & u \end{pmatrix} \qquad x(t) = e^{Jt}x_0 = \begin{pmatrix} e^{ut} & te^{ut} & \frac{t^2}{2}e^{ut} \\ 0 & e^{ut} & te^{ut} \\ 0 & 0 & e^{ut} \end{pmatrix} x_0.$$

IV. $a, b, u \in \mathbb{R}$ and $b \neq 0$:

$$J = \begin{pmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & u \end{pmatrix} \qquad x(t) = e^{Jt}x_0 = \begin{pmatrix} e^{at}\cos(bt) & -e^{at}\sin(bt) & 0 \\ e^{at}\sin(bt) & e^{at}\cos(bt) & 0 \\ 0 & 0 & e^{ut} \end{pmatrix} x_0.$$

An interesting special case is where $a = 0$.

We will take a look at examples of all of these in class.

NONHOMOGENEOUS SYSTEMS

Proposition. Let $A \in M_n(F)$ and consider the system

$$x'(t) = Ax(t) + b(t)$$

where $t \mapsto b(t) \in F^n$ is continuous. The solution with initial condition x_0 is

$$x(t) = e^{At}x_0 + e^{At}\int_{s=0}^{t} e^{-As}b(s)\,ds.$$

The solution is unique.

Proof. Defining $x(t)$ as above, use the product rule and the fundamental theorem of calculus to see

$$\begin{aligned} x'(t) &= \left(e^{At}x_0\right)' + \left(e^{At}\right)'\int_{s=0}^{t} e^{-As}b(s)\,ds + e^{At}\left(\int_{s=0}^{t} e^{-As}b(s)\,ds\right)' \\ &= Ae^{At}x_0 + Ae^{At}\int_{s=0}^{t} e^{-As}b(s)\,ds + e^{At}e^{-At}b(t) \\ &= Ae^{At}x_0 + Ae^{At}\int_{s=0}^{t} e^{-As}b(s)\,ds + e^{At}e^{-At}b(t) \end{aligned}$$

$$
\begin{aligned}
&= A\left(e^{At}x_0 + e^{At}\int_{s=0}^{t} e^{-As}b(s)\,ds\right) + b(t)\\
&= Ax(t) + b(t).
\end{aligned}
$$

Uniqueness of the solution will be a homework problem. $\square$

Week 5, Wednesday: Nonhomogeneous equations

NONHOMOGENEOUS SYSTEMS

Proposition. Let $A \in M_n(F)$ and consider the system

$$x'(t) = Ax(t) + b(t)$$

where $t \mapsto b(t) \in F^n$ is continuous. The solution with initial condition x_0 is

$$x(t) = e^{At}x_0 + e^{At}\int_{s=0}^{t} e^{-As}b(s)\,ds.$$

The solution is unique.

Proof. Given in the last lecture: just take the derivative of the above expression. Uniqueness is a homework problem. □

Note. Our text has references for a system as in the Proposition but for which $A = A(t)$, i.e., A varies with t, too.

Example. Here is an example from our text for an equation modeling a forced harmonic oscillator:

$$x'' = -x + f(t).$$

Writing $x_1 = x$ and $x_2 = x_1'$, we have

$$x_2' = x_1'' = -x + f(t) = -x_1 + f(t).$$

Hence, we consider the system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 + f(t) \end{aligned}$$

or let

$$y := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x \\ x' \end{pmatrix}$$

and consider the system

$$y' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} y + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}.$$

So we apply the proposition with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad b(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}.$$

Thus,

$$e^{At} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix},$$

and

$$\begin{aligned} y(t) &= e^{At} y_0 + e^{At} \int_{s=0}^{t} e^{-As} b(s)\, ds \\ &= \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} y_0 \\ &\quad + \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \int_{s=0}^{t} \begin{pmatrix} \cos(s) & -\sin(s) \\ \sin(s) & \cos(s) \end{pmatrix} \begin{pmatrix} 0 \\ f(s) \end{pmatrix} ds \\ &= \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} y_0 + \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \int_{s=0}^{t} \begin{pmatrix} -f(s)\sin(s) \\ f(s)\cos(s) \end{pmatrix} ds. \end{aligned}$$

The initial condition is $y_0 = (x_1(0), x_2(0)) = (x(0), x'(0))$. We take the first component of the above 2×1 matrix to get the solution:

$$\begin{aligned} x(t) &= x(0)\cos(t) + x'(0)\sin(t) \\ &\quad + \cos(t)\left(-\int_{s=0}^{t} f(s)\sin(s)\, ds\right) + \sin(t)\left(\int_{s=0}^{t} f(s)\cos(s)\, ds\right) \\ &= x(0)\cos(t) + x'(0)\sin(t) + \int_{s=0}^{t} f(s)\left(-\cos(t)\sin(s) + \sin(t)\cos(s)\right) ds. \end{aligned}$$

Now use the sum formula

$$\sin(\theta + \psi) = \cos(\theta)\sin(\psi) + \cos(\psi)\sin(\theta)$$

with $\theta = t$ and $\psi = -s$ to get

$$x(t) = x(0)\cos(t) + x'(0)\sin(t) + \int_{s=0}^{t} f(s)\sin(t - s)\, ds.$$

For a special case, suppose that $f(s) = \cos(\omega t)$. The solution is then

$$x(t) = x(0)\cos(t) + x'(0)\sin(t) + \int_{s=0}^{t} \cos(\omega s)\sin(t-s)\,ds.$$

To integrate this, note that

$$\begin{aligned}\sin(\theta+\psi) + \sin(\theta-\psi) &= \cos(\theta)\sin(\psi) + \cos(\psi)\sin(\theta)\\ &\quad - \cos(\theta)\sin(\psi) + \cos(\psi)\sin(\theta)\\ &= 2\cos(\psi)\sin(\theta).\end{aligned}$$

Therefore,

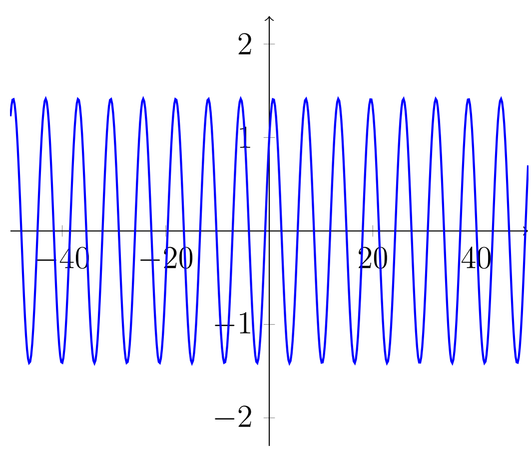
$$\cos(\psi)\sin(\theta) = \frac{1}{2}\left(\sin(\theta+\psi) + \sin(\theta-\psi)\right).$$

It follows that

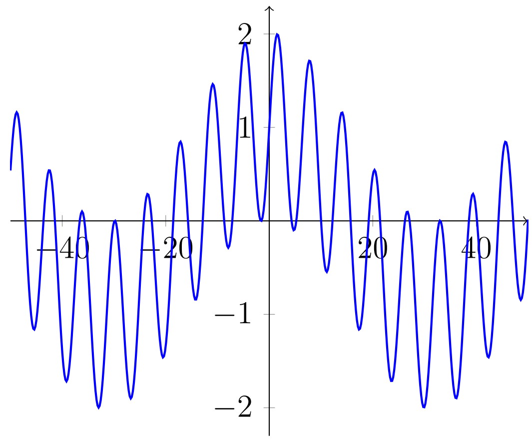
$$\begin{aligned}\int_{s=0}^{t} \cos(\omega s)\sin(t-s)\,ds &= \frac{1}{2}\int_{s=0}^{t} \sin(t+(\omega-1)s) + \sin(t-(\omega+1)s)\,ds\\ &= \frac{1}{2}\left(-\frac{\cos(t+(\omega-1)s)}{\omega-1} + \frac{\cos(t-(\omega+1)s)}{\omega+1}\right)\Bigg|_{s=0}^{t}\\ &= \frac{\cos(\omega t) - \cos(t)}{1-\omega^2}.\end{aligned}$$

So the solution is

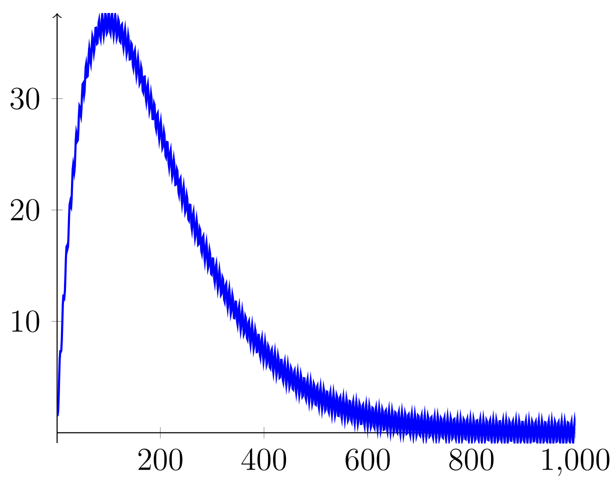
$$x(t) = x(0)\cos(t) + x'(0)\sin(t) + \frac{\cos(\omega t) - \cos(t)}{1-\omega^2}.$$



Unforced: $x(0) = x'(0) = 1$, $f(t) = 0$



$x(0) = x'(0) = 1$, $\omega = 0.1$



$x(0) = x'(0) = 1, \; f(t) = te^{-0.01t}$

Week 5, Friday: Higher-order homogeneous linear equations with constant coefficients

Consider the 4th order linear homogeneous equation with constant coefficients:

$$y^{(iv)} + a_3 y''' + a_2 y'' + a_1 y' + a_0 y = 0 \tag{15.1}$$

with initial condition $y^{(i)}(0) = b_i$ for $i = 0, 1, 2, 3$.

Let $x_i = y^{(i-1)}$ for $i = 1, 2, 3, 4$, and let $x(t) = (x_1(t), \ldots, x_4(t))$. Use equation (15.1) to create a linear system:

$$x' = Ax$$

with initial condition $x(0)$.

PROBLEM 1. What is the 4×4 matrix A? And what is $x(0)$ in terms of y?

PROBLEM 2. The solution to the above system is $x(t) = e^{At} x_0$. Suppose you have calculated e^{At}. How do read off the solution to our original equation (15.1)?

According to the recipe we learned during the first couple of weeks of class, to solve equation (15.1), we first consider its characteristic polynomial $P(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$. We would like to compare $P(x)$ to $p_A(x) := \det(A - xI_4)$, the characteristic polynomial for the matrix A.

PROBLEM 3. Compute $\det(A - xI_4)$ by first performing the following column operations (which don't affect the value of the determinant): add x times the second column to the first column, then add x^2 times the third column to the first column, then add x^3 times the fourth column to the first column. (i) What is the result? The first column should consist of zeros except for the last entry. (ii) What is this last entry? (iii) Compute the determinant by expanding along the first column. What do you get? (iv) What would you get if instead of starting with a 4-th degree equation, we started with an n-th degree equation?

Let λ be an eigenvalue for A, and consider the corresponding eigenspace,

$$E_\lambda = \left\{ v \in F^4 : Av = \lambda v \right\}.$$

PROBLEM 4. Prove that

$$E_\lambda = \operatorname{Span}\left\{ (1, \lambda, \lambda^2, \lambda^3) \right\}.$$

Thus, $\dim E_\lambda = 1$, i.e., the geometric multiplicity of λ is 1. (Hint: let $v = (v_1, \ldots, v_n)$ and then compare components on both sides of the equation $Av = \lambda v$.)

Suppose that A has k distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ over $\mathbb{C}$ with multiplicities $m_1, \ldots, m_k$, respectively. So the characteristic polynomial factors as $p_A(x) = \prod_{i=1}^{k} (\lambda_i - x)^{m_i}$.

Problem 5. Why do we know that the Jordan form for A over the complex numbers is

$$\begin{pmatrix} J_{m_1}(\lambda_1) & & & \\ & J_{m_2}(\lambda_2) & & \text{\huge 0} \\ & & \ddots & \\ \text{\huge 0} & & & J_{m_k}(\lambda_k) \end{pmatrix}?$$

Problem 6. Define the *basic functions* for equation (15.1) to be

$$\left\{ t^j e^{\lambda_i t} : 0 \le j < m_i, 1 \le i \le k \right\}.$$

Prove that every solution to equation (15.1) is a linear combination of these basic functions.

We would like to show that each of the basic functions is a solution to equation (15.1). Consider the differential operator $D := \frac{d}{dt}$. We can write equation (15.1) as

$$P(D)y = 0$$

where $P(D) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$. Further, we know (why?) that

$$P(D) = \prod_{i=1}^{k}(D - \lambda_i)^{m_i}.$$

PROBLEM 7.

(a) Prove by induction that for every sufficiently differentiable function $f(t)$, we have

$$(D - \lambda)^k(f(t)e^{\lambda t}) = e^{\lambda t}D^k f(t)$$

for $k \geq 0$.

(b) Show that it follows that

$$P(D)(f(t)e^{\lambda t}) = e^{\lambda t}P(D+\lambda)f(t).$$

(c) Use these results to show that each basic function is a solution to equation (15.1).

Finally, we'd like to show that each solution to equation (15.1) with the given initial condition is a *unique* linear combination of the basic functions. To do so, list the basic functions in some order f_1, f_2, f_3, f_4. For each $\alpha = (\alpha_1, \alpha_, \alpha_3, \alpha_4) \in \mathbb{C}^4$, consider the solution

$$s_\alpha(t) = \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 + \alpha_4 f_4,$$

and, in general, define

$$\begin{aligned} \phi_\alpha \colon \mathbb{C}^4 &\to \mathbb{C}^4 \\ \alpha &\mapsto (s(0), s'(0), s''(0), s'''(0)). \end{aligned}$$

It's clear that ϕ is linear (since differentiation and evaluation are linear).

Problem 8. You have already shown that is ϕ surjective. How? Why does it then follow that ϕ injective? How does this prove uniqueness?

Week 6, Monday: Higher-order homogeneous linear equations with constant coefficients

n-th order linear homogeneous equations revisited

Consider the homogeneous linear equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \tag{16.1}$$

with initial condition $y^{(i)}(0) = b_i$ for $i = 0, 1, \ldots, n-1$. Recall the method of solution introduced during the first two weeks of class. First we factor the characteristic polynomial

$$P(x) = \prod_{i=1}^{k}(x - \lambda_i)^{m_i},$$

where the λ_i are distinct. We claimed that the most general solution was an arbitrary linear combination of the basic functions

$$e^{\lambda_i t},\ te^{\lambda_i t},\ \ldots,\ t^{m_i-1}e^{\lambda_i t}$$

for $i = 1, \ldots, k$. We also claimed that for each initial condition, there would be a unique solution. We now want to justify those claims.

Define

$$x_1 := y,\ x_2 := y',\ x_3 := y'',\ \ldots\ ,\ x_n := y^{(n-1)}.$$

We get a corresponding matrix equation

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ \vdots \\ x_{n-1}' \\ x_n' \end{pmatrix} = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & \text{\Large 0} & \\ & & 0 & 1 & & \\ & \text{\Large 0} & & \ddots & \ddots & \\ & & & & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-2} & -a_{n-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}.$$

Let A denote the $n \times n$ matrix above, and we can consider the differential equation $x' = Ax$ with initial condition $x_0 = x(0) = (y(0), y'(0), \dots, y^{(n-1)}(0)) = (b_0, b_1, \dots, b_{n-1})$. The solution is

$$x(t) = e^{At} x_0,$$

and the first component of this solution is $x_1(t) = y(t)$, the solution to the original homogeneous system.

Proposition. Let $P(x) = \sum_{i=0}^{n} a_i x^i$ be the characteristic polynomial for the linear homogeneous equation (16.1). Then the characteristic polynomial for A is

$$p_A(x) := \det(A - xI_n) = (-1)^{n+1} P(x).$$

Proof. We have

$$A - xI_n A = \begin{pmatrix} -x & 1 & & & & \\ & -x & 1 & & \Large 0 & \\ & & -x & 1 & & \\ \Large 0 & & & \ddots & \ddots & \\ & & & & -x & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-2} & -x - a_{n-1} \end{pmatrix}$$

Multiply the second column by x and add it to the first column; then multiply the third column by x^2 and add it to the first column; and so on. This does not affect the determinant, and the rows of the first column are all 0 now except the last, which is

$$-a_0 - a_1 x - a_2 x^2 - \cdots - a_{n-2} x^{n-2} - (x + a_{n-1}) x^{n-1} = -P(x).$$

It follows that

$$\det(A - xI_n) = \det \begin{pmatrix} 0 & 1 & & & & \\ 0 & -x & 1 & & \Large 0 & \\ 0 & & -x & 1 & & \\ \vdots & \Large 0 & & \ddots & \ddots & \\ 0 & & & & -x & 1 \\ -P(x) & -a_1 & -a_2 & \dots & -a_{n-2} & -a_{n-1} - x \end{pmatrix}$$

Expand along the first column to get the result. $\square$

In order to solve the system, we are interested in the Jordan form for A. So we think about this next.

Proposition. Let λ be an eigenvalue for A. Then the corresponding eigenspace is

$$E_\lambda = \mathrm{Span}\{(1, \lambda, \lambda^2, \dots, \lambda^{n-1})\}$$

and is, hence, one-dimensional. So the geometric multiplicity of each eigenvalue for A is 1.

Proof. Suppose that $Av = \lambda v$ where $v = (v_1, \dots, v_n)$. Note that

$$Av = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & \ddots & \ddots & \\ & & & & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-2} & -a_{n-1} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{n-1} \\ v_n \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{n-1} \\ v_n \end{pmatrix} = \lambda v$$

says $v_2 = \lambda v_1,\ v_3 = \lambda v_2,\ \dots\ ,\ v_n = \lambda v_{n-1}$. Thus,

$$\begin{aligned} v &= (v_1, \lambda v_1, \lambda^2 v_1, \dots, \lambda^{n-1} v_1) \\ &= v_1(1, \lambda, \lambda^2, \dots, \lambda^{n-1}). \end{aligned}$$

□

Corollary. Suppose that A has distinct eigenvalues $\lambda_1, \dots, \lambda_k$ (over $\mathbb{C}$) with algebraic multiplicities, $m_1, \dots, m_k$, respectively, so the its characteristic polynomial is

$$p_A(x) = \prod_{i=1}^{k} (\lambda_i - x)^{m_i}.$$

Then the Jordan form for A is

$$\begin{pmatrix} J_{m_1}(\lambda_1) & & & \\ & J_{m_2}(\lambda_2) & & \text{\Large 0} \\ & \text{\Large 0} & \ddots & \\ & & & J_{m_k}(\lambda_k) \end{pmatrix}.$$

Proof. This follows immediately from the preceding Proposition. The diagonal of the Jordan form consists of the eigenvalues of A, repeated according to multiplicities. For each Jordan block, there is a corresponding eigenvector for A (and several generalized eigenvectors). If there where more than one Jordan block for a particular eigenvalue λ, there would be more than one linearly independent eigenvector for λ, and we've just seen that this cannot happen—each eigenspace has dimension 1. □

Theorem. Suppose the roots for the characteristic polynomial for equation (16.1) or, equivalently, the eigenvalues for A are $\lambda_1, \ldots, \lambda_k$ with multiplicities $m_1, \ldots, m_k$, respectively. Every solution to equation (16.1) (with a given initial condition) is a unique linear combination of the *basic functions*

$$\left\{t^j e^{\lambda_i t} : 0 \leq j < m_i, 1 \leq i \leq k\right\}, \tag{16.2}$$

and each linear combination of these functions is a solution for some initial condition.

Proof. There are three parts to this proof: (i) show each solution is a linear combination of the basic functions; (ii) show each basic function satisfies the differential equation (16.1); and (iii) show the basic equations are linearly independent.

(i) The solution to equation (16.1) is the first component of $e^{At}x_0$. Letting $P^{-1}AP = J$ be the Jordan form for A, the solution is

$$y(t) = e^{At}x_0 = Pe^{Jt}P^{-1},$$

and hence, is a linear combination of the entries of e^{Jt}. The result then follows from the previous corollary recalling that

$$e^{J_{m_i}(\lambda_i t)} = e^{\lambda_i t}\begin{pmatrix} 1 & t & \frac{t^2}{2!} & \ldots & \ldots & \frac{t^{m_i-1}}{(m_i-1)!} \\ 0 & 1 & t & \ldots & \ldots & \frac{t^{k-2}}{(m_i-2)!} \\ 0 & 0 & 1 & \ldots & \ldots & \frac{t^{k-3}}{(m_i-3)!} \\ & \ddots & & \vdots & \vdots & \vdots \\ 0 & \ldots & \ldots & 0 & 1 & t \\ 0 & \ldots & \ldots & \ldots & 0 & 1 \end{pmatrix}.$$

(ii) Consider the differential operator $D := \frac{d}{dt}$. We can write equation (16.1) as

$$P(D)y = 0$$

where $P(D) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. We are given that

$$P(D) = \prod_{i=1}^{k} (D - \lambda_i)^{m_i}.$$

That the basic functions satisfy the differential equation $P(D)y = 0$ is left as homework. It follows from two facts (which are part of the homework problem):

1. $(D - \alpha)(D - \beta)f(t) = (D - \beta)(D - \alpha)f(t)$ for every sufficiently differentiable function $f(t)$ and pair of constants α and β.

2. $P(D)(f(t)e^{\lambda t}) = e^{\lambda t} P(D+\lambda)(f(t))$ for every sufficiently differentiable function $f(t)$ and constant λ.

(iii) For uniqueness, list the n functions in (16.2) in some order $f_1, \ldots, f_n$, and consider the mapping $\phi\colon \mathbb{C}^n \to \mathbb{C}^n$ defined as follows: for each $(\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$, consider the solution

$$s_\alpha(t) = \alpha_1 f_1 + \ldots \alpha_n f_n,$$

and let

$$\phi(\alpha_1, \ldots, \alpha_n) := (s_\alpha(0), s'_\alpha(0), \ldots, s_\alpha^{(n-1)}(0)) \in \mathbb{C}^n.$$

Since taking differentiation and evaluation are both linear operations, ϕ is linear. It is surjective since we know from part (ii) that we can find a solution as a linear combination of $f_1, \ldots, f_n$ for each initial condition. Since ϕ is linear and has rank 4, i.e., $\dim(\operatorname{im} \phi) = 4$, the rank-nullity theorem says that the kernel of ϕ is trivial. So ϕ is injective. Now take any two solutions that satisfy the same initial conditions. Each of these solutions is a linear combination of the basic functions, so they have the form s_α and s_β for some $\alpha, \beta \in \mathbb{C}^n$. Since they satisfy the same initial condition, we have $\phi(\alpha) = \phi(\beta)$. Since ϕ is injective, we have $\alpha = \beta$.

Week 6, Wednesday: Existence and uniqueness for non-linear systems

NONLINEAR SYSTEMS

Let $E \subseteq \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$ and let $C(E)$ denote the vector space of continuous functions of the form $E \to \mathbb{R}^n$. Given $f \in C(E)$, we are now interested in solutions to the differential equation

$$x' = f(x). \tag{17.1}$$

The function f is a vector field in $\mathbb{R}^n$ defined on E. We have just finished studying the linear case of this problem, i.e., in which $f(x) = Ax$ for some $A \in M_n(\mathbb{R}^n)$ and are now particularly interested in the case where f is no longer a linear function.

A *solution* to equation (17.1) on an interval I is a function $x\colon I \to E \subseteq \mathbb{R}^n$ such that

$$x'(t) = f(x(t))$$

for all $t \in I$. Given $t_0 \in I$ with $x(t_0) = x_0 \in E$, we say the solution satisfies the *initial value problem*

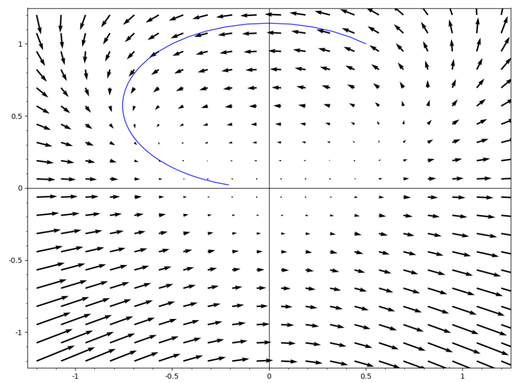
$$\begin{aligned} x' &= f(x) \\ x(t_0) &= x_0 \end{aligned}$$

on I.

Example. Consider the (non-linear) system

$$\begin{aligned} x' &= x^2 - y \\ y' &= xy \end{aligned}$$

with initial value $(x(0), y(0)) = (0.5, 1)$. So in this case, the relevant vector field is $f(x, y) = (x^2 - y, xy)$. Here is a plot of the vector field and the solution to the initial value problem:



Note that this system displays behavior one would not see in the linear case.

The systems we are studying are called *autonomous* since f is a function of $x \in \mathbb{R}^n$ and not t. However, a *nonautonomous* system

$$x' = g(x, t)$$

can be converted to an autonomous system by letting $x_{n+1} := t$ and $x'_{n+1} = 1$.

Goals. Our first main goal is to find conditions under which the initial value problem for equation (17.1) has a unique solution. After that, we'll discuss how solutions change if f changes a small amount and discuss the size of the interval on which a solution exists.

New behavior. In the linear case, $x' = Ax$ and $x(0) = x_0$, we saw that there is always a unique solution. That's no longer generally true in the nonlinear case. For instance, the following initial value problem

$$\begin{aligned} x' &= 3x^{2/3} \\ x(0) &= 0 \end{aligned}$$

has two solutions: $x(t) = 0$ and $x(t) = t^3$. We'll see that the source of non-uniqueness here is that $f(x) = 3x^{2/3}$ is not continuously differentiable: $f'(x) = 2x^{-1/3}$, which is not continuous at 0.

Even if f' is continuous everywhere, the solution may only exist on subintervals of the real line, again unlike the linear situation. For example, consider the system

$$\begin{aligned} x' &= x^2 \\ x(0) &= 1. \end{aligned}$$

The solution is
$$x(t) = \frac{1}{1-t}$$
but is only defined on the interval $(-\infty, 1)$. The solution blows up as $t \to 1^-$.

Key idea. We have solved the initial value problem for equation (17.1) if we can find a continuous function $x(t)$ satisfying
$$x(t) = x_0 + \int_{s=0}^{t} f(x(s))\, ds$$
for all $t \in [-a, a]$ for some $a > 0$.

Check: First, by the fundamental theorem of calculus
$$x'(t) = (x_0)' + \left(\int_{s=0}^{t} f(x(s))\, ds\right)' = 0 + f(x(t)) = f(x(t)).$$
Next,
$$x(0) = x_0 + \int_{s=0}^{0} f(x(s))\, ds = x_0.$$
The *method of successive approximations* attempts to create a sequence of functions $(u_k(t))_{k\geq 0}$ converging to a solution:
$$\begin{aligned} u_0(t) &:= x_0 \\ u_{k+1}(t) &:= x_0 + \int_{s=0}^{t} f(u_k(s))\, ds, \quad \text{for } k \geq 0. \end{aligned}$$

Example. Consider the initial value problem
$$x' = xt, \quad x(0) = 1.$$
This is an autonomous equation, so we first convert it to a nonautonomous system by letting $x_1 = x$ and $x_2 = t$. The system becomes
$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_1 x_2 \\ 1 \end{pmatrix} =: f(x_1, x_2)$$
with initial condition $x_1(0) = x(0) = 1$ and $x_2(0) = 0$ (since $x_2 = t$).

Apply the method of successive approximations starting with
$$u_0(t) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We get

$$\begin{aligned}
u_1(t) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{s=0}^{t} f(u_0(s))\,ds \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{s=0}^{t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} ds \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} c \\ s \end{pmatrix}\Bigg|_{s=0}^{t} \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ t \end{pmatrix}.
\end{aligned}$$

Next,

$$\begin{aligned}
u_2(t) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{s=0}^{t} f(u_1(s))\,ds \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{s=0}^{t} f(1, s)\,ds \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{s=0}^{t} \begin{pmatrix} s \\ 1 \end{pmatrix} ds \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} s^2/2 \\ s \end{pmatrix}\Bigg|_{s=0}^{t} \\
&= \begin{pmatrix} 1 + t^2/2 \\ t \end{pmatrix}.
\end{aligned}$$

Next,

$$\begin{aligned}
u_3(t) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{s=0}^{t} f(u_2(s))\,ds \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{s=0}^{t} f(1 + s^2/2, s)\,ds \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{s=0}^{t} \begin{pmatrix} s + s^3/2 \\ 1 \end{pmatrix} ds \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} s^2/2 + s^2/(2 \cdot 4) \\ s \end{pmatrix}\Bigg|_{s=0}^{t}
\end{aligned}$$

$$= \begin{pmatrix} 1 + t^2/2 + t^4/(2 \cdot 4) \\ t \end{pmatrix}.$$

Similarly,

$$u_4 = \begin{pmatrix} 1 + t^2/2 + t^4/(2 \cdot 4) + t^6/(2 \cdot 4 \cdot 6) \\ t \end{pmatrix},$$

and so on. Recall that $x_1 = x$, and x is the function we are trying to find. Thus, we are interested in the limit of the first components of the u_k. The method of successive approximations is delivering

$$\begin{aligned} x(t) &= 1 + \frac{t^2}{2} + \frac{t^4}{2 \cdot 4} + \frac{t^6}{2 \cdot 4 \cdot 6} + \frac{t^8}{2 \cdot 4 \cdot 6 \cdot 8} + \dots \\ &= 1 + \frac{t^2}{2} + \frac{t^4}{(1 \cdot 2)2^2} + \frac{t^6}{(1 \cdot 2 \cdot 3)2^3} + \frac{t^8}{(1 \cdot 2 \cdot 3 \cdot 4)2^3} + \dots \\ &= 1 + \frac{t^2}{2} + \frac{1}{2!}\left(\frac{t^2}{2}\right)^2 + \frac{1}{3!}\left(\frac{t^2}{2}\right)^3 + \frac{1}{4!}\left(\frac{t^2}{2}\right)^4 + \dots \\ &= e^{t^2/2}, \end{aligned}$$

which converges, and it's easy to check that it satisfies the original initial value problem:

$$x'(t) = \left(e^{t^2/2}\right)' = te^{t^2/2} = x(t)t,$$

and $x(0) = 1$.

Of course, we could have solved the equation through separation of variables:

$$x' = xt \quad \Rightarrow \quad \int \frac{dx}{x} = \int t\,dt \quad \Rightarrow \quad \ln(x) = t^2/2 + c.$$

Then $x(0) = 1$ implies $c = 0$. Exponentiate:

$$\ln(x) = t^2/2 \quad \Rightarrow \quad x = e^{t^2/2}.$$

Fixed points. Consider the operator on functions, $u \to T(u)$ given by

$$T(u)(t) := x_0 + \int_{s=0}^{t} f(u(s))\,ds$$

In the case we just considered, with $x_0 = (1,0)$ and $f(x_1,x_2) = (x_1x_2, 1)$, the method of successive iterations produced the function $u(t) = (e^{t^2/2}, t)$. This function u is a *fixed point* for the operator T:

$$\begin{aligned}
T(u(t)) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{s=0}^{t} f(u(s))\,ds \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{s=0}^{t} f(e^{s^2/2}, s)\,ds \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{s=0}^{t} \begin{pmatrix} se^{s^2/2} \\ 1 \end{pmatrix} ds \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} e^{s^2/2} \\ s \end{pmatrix}\Bigg|_{s=0}^{t} \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} e^{t^2/2} - 1 \\ t \end{pmatrix} \\
&= \begin{pmatrix} e^{t^2/2} \\ t \end{pmatrix} \\
&= u(t).
\end{aligned}$$

Next step. We have seen that the method of successive approximations amounts to iterating a operator on a space of functions and converging to a fixed point for that operator. Our next step is to consider this situation a little more generally. Let X be a space in which convergence makes sense, and consider a mapping $T\colon X \to X$. We would like to know conditions under which iterates of a point $x_0 \in X$ under T will converge to a point $\tilde{x} \in X$ that is fixed under T, i.e., such that $T(\tilde{x}) = \tilde{x}$.

Week 6, Friday: Existence and uniqueness for non-linear systems

THE CONTRACTION MAPPING PRINCIPLE

Let $(V, \| \ \|)$ be a normed vector space over $F = \mathbb{R}$ or $\mathbb{C}$. Recall this means that for all $v, w \in V$ and $\alpha \in F$,

1. $\|v\| \geq 0$ with equality if and only if $v = 0$;
2. $\|\alpha v\| = |\alpha|\|v\|$;
3. $\|v + w\| \leq \|v\| + \|w\|$.

If every Cauchy sequence in V converges (in V), then we say V is *complete*, and in that case $(V, \| \ \|)$ is called a *Banach space*. We have already used the fact that $\mathbb{R}^n$ and $\mathbb{C}^n$ are Banach spaces, for example, when considering the convergence of e^{At}. We will soon need to consider a Banach space whose elements consist of potential solutions to systems of differential equations.

Definition. Let $(V, \| \ \|)$ be a Banach space, and let $X \subseteq V$. Let $T \colon X \to X$.

1. A point $u \in X$ is a *fixed point* for T if $T(u) = u$.
2. The function T is a *contraction mapping* if there is a constant $c \in [0, 1) \subset \mathbb{R}$ such that
$$\|T(u) - T(v)\| \leq c\|u - v\|$$
for all $u, v \in X$.

Theorem. Let $(V, \| \ \|)$ be a Banach space, and let $X \subseteq V$ be a closed subset of V (hence, it contains all of its limit points). Suppose that $T \colon X \to X$ is a contraction mapping and fix a constant $c \in [0, 1) \subset \mathbb{R}$ so that

$$\|T(u) - T(v)\| \leq c\|u - v\|$$

for all $u, v \in X$. Then T has a unique fixed point $\tilde{u} \in X$. Let $u_0 \in X$ and consider the sequence of iterates

$$u_0, T(u_0), T^2(u_0), T^3(u_0), \ldots$$

(For example, $T^3(u_0) = T(T(T(u_0)))$.) We have, for all $m \geq 0$,

$$\|\tilde{u} - T^m(u_0)\| \leq \frac{c^m}{1-c}\|T(u_0) - u_0\|.$$

In particular, the sequence of iterates converges to the fixed point, $\tilde{u}$.

Proof. We first show uniqueness. Suppose that $T(u) = u$ and $T(v) = v$. We have

$$\|u - v\| = \|T(u) - T(v)\| \leq c\|u - v\|,$$

which implies

$$(1 - c)\|u - v\| \leq 0.$$

Since $1 - c \geq 0$, it follows that $\|u - v\| = 0$, and hence, $u = v$.

Now take $u_0 \in X$, and define $u_{k+1} := T(u_k)$ for $k \geq 0$. Thus, $u_k = T^k(u_0)$ for all $k \geq 0$. For all pairs of natural numbers $m \leq n$,

$$\begin{aligned}
\|u_n - u_m\| &= \|T(u_{n-1}) - T(u_{m-1})\| \\
&\leq c\|u_{n-1} - u_{m-1}\| \\
&= c\|T(u_{n-2}) - T(u_{m-2})\| \\
&\leq c^2\|u_{n-2} - u_{m-2}\| \\
&\vdots \\
&\leq c^m\|u_{n-m} - u_0\| \\
&= c^m\|(u_1 - u_0) + (u_2 - u_1) + (u_3 - u_2) \cdots + (u_{n-m} - u_{n-m-1})\| \\
&\leq c^m\left(\|u_1 - u_0\| + \|u_2 - u_1\| + \|u_3 - u_2\| + \cdots + \|u_{n-m} - u_{n-m-1}\|\right) \\
&\leq c^m\left(\|u_1 - u_0\| + c\|u_1 - u_0\| + c^2\|u_1 - u_0\| + \cdots + c^{n-m-1}\|u_1 - u_0\|\right) \\
&= c^m\|u_1 - u_0\|\left(1 + c + c^2 + \cdots + c^{n-m-1}\right) \\
&= c^m\frac{1 - c^{n-m}}{1 - c}\|u_1 - u_0\| \\
&\leq \frac{c^m}{1-c}\|u_1 - u_0\|.
\end{aligned}$$

Given any $\varepsilon > 0$, we then see that by choosing N sufficiently large, it follows that if $m, n \geq N$, then $\|u_n - u_m\| < \varepsilon$. So the sequence of iterates, $(u_k)_{k\geq 0}$ is Cauchy. Since V is a Banach space, the sequence converges to some $\tilde{u}$, and since X is closed, $\tilde{u} \in X$. Since T is a contraction mapping, it's continuous (exercise), and therefore commutes with limits:

$$\lim_{k\to\infty} T(u_k) = T(\lim_{k\to\infty} u_k) = T(\tilde{u}).$$

On the other hand, by definition of the u_k, we have

$$\lim_{k\to\infty} T(u_k) = \lim_{k\to\infty} u_{k+1} = \lim_{k\to\infty} u_k = \tilde{u}.$$

This shows $T(\tilde{u}) = \tilde{u}$, i.e., $\tilde{u}$ is the unique fixed point of T.

Finally, in our calculation above, we saw that for all $m \leq n$,

$$\|u_n - u_m\| = \|T^n(u_0) - T^m(u_0)\| \leq \frac{c^m}{1-c}\|u_1 - u_0\| \leq \frac{c^m}{1-c}\|T(u_0) - u_0\|.$$

Since the norm function and T are continuous, they both commute with limits. Therefore, taking the limit as $n \to \infty$ on both sides of the above inequality yields

$$\|\tilde{u} - T^m(u_0)\| \leq \frac{c^m}{1-c}\|T(u_0) - u_0\|.$$

$\square$

Method of successive approximations. We are interested in applying the contraction mapping principle to the operator

$$T(u) = x_0 + \int_{s=0}^{t} f(u(s))\, ds,$$

discussed in the previous lecture. So we need to find the appropriate Banach space and find conditions under which T is a contraction mapping.

Definition. If $I \subset \mathbb{R}$ is a closed bounded interval, let $C(I)$ denote the $\mathbb{R}$-vector space of continuous functions on $I \to \mathbb{R}^n$ (where n is fixed). For each $u \in C(I)$, define

$$\|u\| := \sup_{t\in I} |u(t)| = \max_{t\in I} |u(t)|.$$

(The last equality is due to the fact that the continuous image of a compact set is compact—a generalization of the extreme value theorem of one-variable calculus.) Geometrically, $\|u\|$ is the maximum distance from the origin reached by $u(t)$.

Proposition. $(C(I), \|\ \|)$ is a Banach space.

Proof. Math 321. $\square$

Thus, the method of successive approximations is an operator

$$T\colon C(I) \to C(I)$$

on the Banach space of continuous functions on I. Under what conditions is it a contraction mapping? We have

$$\begin{aligned}
|(Tu)(t) - (Tv)(t)| &= \left| \left(x_0 + \int_{s=0}^{t} f(u(s))\, ds \right) - \left(x_0 + \int_{s=0}^{t} f(v(s))\, ds \right) \right| \\
&= \left| \int_{s=0}^{t} f(u(s)) - f(v(s))\, ds \right| \\
&\leq \int_{s=0}^{t} |f(u(s)) - f(v(s))|\, ds \\
&\leq t \max_{s\in[0,t]} \{ |f(u(s)) - f(v(s))| \}.
\end{aligned}$$

From this, we can see two things that will help to control the size of $|T(u) - T(v)|$: first, restrict to a small enough region around x_0 so that f does not vary much on that region, and second, make the interval in which t varies small. We address the first problem below by considering the derivative of f.

Week 7, Monday: Existence and uniqueness for non-linear systems

FUNDAMENTAL EXISTENCE AND UNIQUENESS THEOREM

Our goal is to apply the contraction mapping principle to the operator

$$\begin{aligned} T\colon C(I) &\to C(I) \\ u &\mapsto x_0 + \int_{s=0}^{t} f(u(s))\, ds \end{aligned}$$

in order to prove the fundamental existence and uniqueness theorem for ordinary differential equations.

Derivative review. Let $E \subseteq \mathbb{R}^n$ be an open set. Recall from vector calculus that the derivative of a function $f\colon E \to \mathbb{R}^n$ at a point $p \in E$ is a linear function

$$Df_p\colon \mathbb{R}^n \to \mathbb{R}^n$$

approximating f near p:

$$f(p+h) \approx f(p) + Df_p(h)$$

for small h. Its corresponding matrix is the Jacobian matrix for f at p, whose j-th column is the j-th partial of f (measuring how f is changing in the j-th coordinate direction):

$$\frac{\partial f}{\partial x_j}(p) = \begin{pmatrix} \frac{\partial f_1}{\partial x_j}(p) \\ \frac{\partial f_2}{\partial x_j}(p) \\ \vdots \\ \frac{\partial f_n}{\partial x_j}(p) \end{pmatrix}.$$

We say $f\colon E \to \mathbb{R}^n$ is *continuously differentiable* if it is differentiable at all points in E and the mapping

$$E \to \mathcal{L}(\mathbb{R}^n)$$

$$p \mapsto Df_p$$

is continuous.

Explanation: First, $\mathcal{L}(\mathbb{R}^n)$ denotes the vector space of linear functions from $\mathbb{R}^n$ to itself. Second, to talk about continuity we define a norm on $\mathcal{L}(\mathbb{R}^n)$: for $L \in \mathcal{L}(\mathbb{R}^n)$, let

$$\|L\| = \max_{|x| \leq 1} |L(x)|.$$

This is the same as $\|A\|$ if A is the matrix representing L. In that case, since $L(x) = Ax$, the inequality $|Ax| \leq \|A\||x|$ can be written as

$$\|L(x)\| \leq \|L\| \, |x|.$$

A theorem from calculus says that f is continuously differentiable if and only if all of its partials $\partial f_i / \partial x_j$ exist and are continuous. (Also, it turns out that continuity of the partials guarantees that f is differentiable.)

Notation. For an open subset $E \subset \mathbb{R}^n$, we denote the $\mathbb{R}$-vector space of continuously differentiable functions on E by $C^1(E)$.

Lipschitz condition. We now introduce a condition on vector fields that will allow the application of the contraction mapping principle to T.

Definition. Let $E \subseteq \mathbb{R}^n$ be an open subset. Then a function $f \colon E \to \mathbb{R}^n$ is *Lipschitz* if there exists a constant K such that

$$|f(x) - f(y)| \leq K|x - y|$$

for all $x, y \in E$. On the other hand, f is *locally Lipschitz* on E if for each $x_0 \in E$, there exists $\varepsilon > 0$ and a constant K_{x_0} such that

$$|f(x) - f(y)| \leq K_{x_0}|x - y|$$

for all

$$x, y \in N_\varepsilon(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < \varepsilon\}.$$

Proposition. If $f \in C^1(E)$, then f is locally Lipschitz.

Proof. Let $x_0 \in E$. Since E is open it contains an open ball about x_0, i.e., there exists $\eta > 0$ such that $N_\eta(x_0) \subset E$. Define $\varepsilon := \eta/2$ and consider the closed ball

$$B := B_\varepsilon(x_0) := \overline{N_\varepsilon(x_0)} := \{x \in \mathbb{R}^n : |x - x_0| \leq \varepsilon\}.$$

Let

$$K_{x_0} := \max_{x \in B} \|Df_x\|.$$

The constant K_{x_0} exists since we're assuming Df is continuous ($f \in C^1(E)$). Thus, $x \to Df_x \to \|Df_x\|$, being the composition of continuous functions, is also continuous. Since B is convex, given $x, y \in B$, the line segment joining x to y is contained in B. Hence, it is OK to stick these points into f. Parametrize the line segment by $\phi(s) = x + s(y - x)$ for $s \in [0, 1]$ and consider the composition

$$\begin{aligned} F := f \circ \phi :: [0,1] &\to \mathbb{R}^n \\ s &\mapsto f(x + s(y - x)), \end{aligned}$$

a curve in $\mathbb{R}^n$. By the chain rule,

$$DF_s = Df_{\phi(s)} \circ D\phi_s.$$

Since F is a curve in $\mathbb{R}^n$, its Jacobian matrix at s is a single column vector—the tangent or velocity vector $F'(s)$—and

$$DF_s(t) = tF'(s),$$

a linear function of t (for fixed s). Similarly ϕ_s is a curve in $\mathbb{R}^n$, so its Jacobian matrix is its velocity at time s. It's easy to compute: since $\phi(s) = x + s(y - x)$, its velocity is constant. At any time s, we have $\phi'(s) = y - x$. Thus,

$$D\phi_s(t) = t(y - x).$$

By the chain rule,

$$tF'(s) = DF_s(t) = Df_{\phi(s)}(t(y - x)).$$

Setting $t = 1$, we get

$$F'(s) = Df_{(x+s(y-x))}(y - x) \in \mathbb{R}^n.$$

Since $F(0) = f(x)$ and $F(1) = f(y)$,

$$\begin{aligned} |f(y) - f(x)| &= |F(1) - F(0)| \\ &= \left| \int_{s=0}^{1} F'(s)\, ds \right| \\ &\leq \int_{s=0}^{1} |F'(s)|\, ds \\ &= \int_{s=0}^{1} |Df_{(x+s(y-x))}(y - x)|\, ds \end{aligned}$$

$$\begin{aligned}
&\leq \int_{s=0}^{1} \|Df(x+s(y-x))\| \, |y-x| \, ds \\
&\leq K_{x_0} \int_{s=0}^{1} |y-x| \, ds \\
&= K_{x_0} |y-x|.
\end{aligned}$$

We've shown that f is locally Lipschitz. $\square$

Theorem. (The fundamental existence and uniqueness theorem for nonlinear systems.) Let E be an open subset of $\mathbb{R}^n$ containing x_0, and let $f \in C^1(E)$. Then there exists $a > 0$ such that the initial value problem

$$\begin{aligned}
x' &= f(x) \\
x(0) &= x_0
\end{aligned}$$

has a unique solution $x(t)$ on $[-a, a]$.

Proof. Since $f \in C^1(E)$, there exists an $\varepsilon > 0$ such that $N_\varepsilon(x_0) \subseteq E$, the open ball of radius ε centered at x_0, and there exists a constant K_{x_0} such that

$$|f(x) - f(y)| \leq K_{x_0} |x - y|$$

for all x, y in $N_\varepsilon(x_0)$. By replacing ε by $\varepsilon/2$, we may assume

$$|f(x) - f(y)| \leq K_{x_0} |x - y|$$

for all x, y in

$$B := \overline{N_\varepsilon(x_0)} := \{x \in \mathbb{R}^n : |x - x_0| \leq \varepsilon\} \subset E.$$

(The point here is to get the Lipschitz condition to hold on a closed bounded ball rather then on the open ball, $N_\varepsilon(x_0)$, in preparation for an application of the extreme value theorem, below.)

Let $I = [-a, a]$ where $a > 0$ is a constant to be determined later, and define

$$X := \{u \in C(I) \colon \|u - x_0\| \leq \varepsilon\},$$

considering $x_0 \in C(I)$ as the constant function $t \mapsto x_0$ for all $t \in I$. This means that for $u \in X$, we have

$$\max_{t \in I} |u(t) - x_0| \leq \varepsilon.$$

In particular, $u(t) \in B \subset E$ for all $t \in I$. Note that B is a subset of $E \subseteq \mathbb{R}^n$ and X is a subset of the function space $C(I)$ of continuous functions $I \to \mathbb{R}^n$. If $u \in X$, then $u(t) \in B$ for all $t \in I$.

Our goal is to show that a can be taken small enough so that (i) $T(u) \in X$ for all $u \in X$, i.e., so that $T\colon X \to X$, and so that (ii) $T\colon X \to X$ is a contraction mapping.

For (i), since B is closed and bounded, we can define

$$M = \max_{x \in B} |f(x)|.$$

Suppose that $0 < a < \frac{\varepsilon}{M}$. Then for $u \in X$ and $t \in I$,

$$\begin{aligned}
|T(u)(t) - x_0| &= \left| \left(x_0 + \int_{s=0}^{t} f(u(s))\, ds \right) - x_0 \right| \\
&= \left| \int_{s=0}^{t} f(u(s))\, ds \right| \\
&\leq \left| \int_{s=0}^{t} |f(u(s))|\ ds \right|.
\end{aligned}$$

If s is in the interval between 0 and t and $u \in X$, it follows that $u(s) \in B$, and hence, $|f(u(s))| \leq M$. Therefore, continuing our calculation,

$$\begin{aligned}
|T(u)(t) - x_0| &= \left| \int_{s=0}^{t} |f(u(s))|\ ds \right| \\
&= \left| \int_{s=0}^{t} M\, ds \right| \\
&= |t|\, M \\
&\leq a\, M \\
&< \frac{\varepsilon}{M}\, M \\
&< \varepsilon.
\end{aligned}$$

Hence,

$$\|T(u) - x_0\| := \max_{t \in I} |T(u)(t) - x_0| < \varepsilon.$$

Therefore $T(u) \in X$. In sum: if $0 < a < \varepsilon/M$, then $T\colon X \to X$.

We now work on (ii): we can take a small enough so that $T\colon X \to X$ is a contraction mapping. Let $u, v \in X$. Then, using the Lipschitz property,

$$\begin{aligned}
|T(u) - T(v)| &= \left| \int_{s=0}^{t} f(u(s)) - f(v(s))\, ds \right| \\
&\leq \left| \int_{s=0}^{t} |f(u(s)) - f(v(s))|\ ds \right| \\
&\leq K_{x_0} \left| \int_{s=0}^{t} |u(s) - v(s)|\ ds \right| \\
&\leq K_{x_0} \left| \int_{s=0}^{t} \max_{c \in I} |u(c) - v(c)|\ ds \right| \\
&= K_{x_0} \left| \int_{s=0}^{t} \|u - v\|\, ds \right| \\
&= K_{x_0}\, |t|\, \|u - v\| \\
&\leq a K_{x_0} \|u - v\|.
\end{aligned}$$

To ensure T is a contraction mapping, take $a = \frac{1}{2K_{x_0}}$ (so that $aK_{x_0} = \frac{1}{2} < 1$).

In total, we have now shown there exists and interval $I = [-a, a]$, a closed ball $X \subset C(I)$ centered at the constant function x_0, such that $T\colon X \to X$ and T is a contraction mapping. It therefore has a unique fixed point $x \in X$. So $x = T(x)$, i.e.,

$$x(t) = T(x)(t) := x_0 + \int_{s=0}^{t} f(x(s))\, ds.$$

By the fundamental theorem of calculus and the fact that $x(0) = x_0$, it follows that x is a solution to the initial value problem

$$\begin{aligned}
x' &= f(x) \\
x(0) &= x_0
\end{aligned}$$

on I. For uniqueness, recall that any solution x on I will be a fixed point for T:

$$T'(x)(t) = \left(x_0 + \int_{s=0}^{t} f(x(s))\, ds \right)' = f(x(t)) = x'(t).$$

so $T(x)$ and x differ by a constant. However $T(x(0)) = x_0 = x(0)$, so that constant is 0. Since every solution is a fixed point of T and contraction mappings have unique fixed points, we are done. $\square$

Week 7, Wednesday: Linearization

Definition. An *equilibrium point* for a system of differential equations in $\mathbb{R}^n$

$$x' = f(x)$$

is a point $p \in \mathbb{R}^n$ such that $f(p) = 0$.

The reason for the terminology is that if p is an equilibrium point then a solution (*the solution* if f is continuously differentiable) with initial condition $x(0) = p$ is the constant solution $x(t) = p$.

We hope to get a qualitative sense of the solutions to our system near an equilibrium point p by replacing the system with a linear approximation:

$$x' = Jf_p$$

where Jf_p is the Jacobian matrix for f at p.

Consider the system of equations

$$\begin{aligned} x' &= (x^2 - 1)y \\ y' &= (1 - y^2)\left(x + \frac{3}{10}y\right). \end{aligned}$$

So in this case $f(x, y) = ((x^2 - 1)y, (1 - y^2)\left(x + \frac{3}{10}y\right))$.

Problem 1. Find all equilibrium points for the system and plot them in the plane.

Problem 2. Compute the Jacobian matrix $Jf_{(x,y)}$ for our f at an arbitrary point (x, y).

Problem 3 For each equilibrium point p, analyze the linear system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = Jf_p \begin{pmatrix} x \\ y \end{pmatrix}$$

by looking at the eigenvalues of Jf_p. Do you get a saddle? A stable focus or node? An unstable focus or node? A center? (See the last page for a quick guide.)

Problem 4. What does the vector field look like along the line $x = 1$ and along the line $x = -1$? What can you say about the special behavior of solutions with an initial condition $(\pm 1, y_0)$? Interpret this geometrically.

Problem 5. What does the vector field look like along the line $y = 1$ and along the line $y = -1$? What can you say about the special behavior of solutions with an initial condition $(x_0, \pm 1)$? Interpret this geometrically.

EQUILIBRIUM POINTS FOR LINEAR SYSTEMS IN $\mathbb{R}^2$

Let $A \in M_2(\mathbb{R})$. Let τ be the trace of A, and let δ be the determinant of A. The characteristic polynomial for A will factor as

$$\begin{aligned} p(x) &= (\lambda_1 - x)(\lambda_2 - x) \\ &= x^2 - (\lambda_1 + \lambda_2)x + \lambda_1\lambda_2 \\ &= x^2 - \tau x + \delta \end{aligned}$$

where λ_1 and λ_2 are the eigenvalues of A. Setting $p(x) = 0$ and solving gives an alternate description of the eigenvalues:

$$x = \frac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2}.$$

If $\delta = 0$, then at least one of the eigenvalues is zero, and we have a **degenerate system**.

$\boxed{\delta = 0}$ **degenerate**.

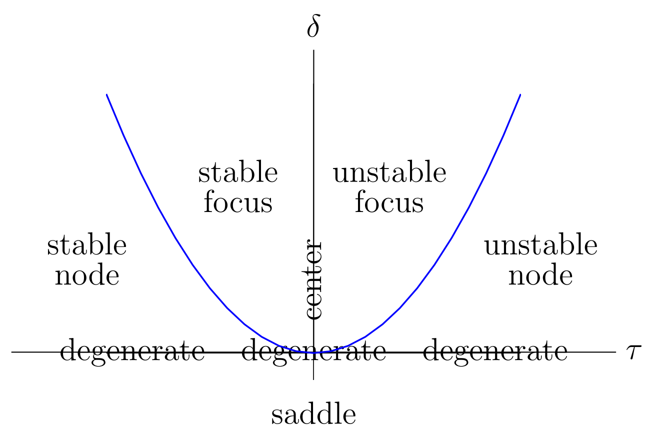
$\boxed{\delta < 0}$ real eigenvalues, opposite signs $\Rightarrow$ **saddle**.

$\boxed{\delta > 0,\ \tau^2 - 4\delta \geq 0}$ real eigenvectors, same signs $\Rightarrow$ node.

$\tau < 0 \Rightarrow$ **stable node**
$\tau > 0 \Rightarrow$ **unstable node**.

$\boxed{\delta > 0,\ \tau^2 - 4\delta < 0}$ nonreal eigenvectors $\Rightarrow$ swirling vector field.

$\tau < 0 \Rightarrow$ **stable focus**
$\tau > 0 \Rightarrow$ **unstable focus**
$\tau = 0 \Rightarrow$ **center**.



Week 7, Friday: Dependence on parameters, maximal interval. Begin stable manifold theorem

Dependence on parameters, maximal interval

Here we mention a couple of fairly immediate refinements of the fundamental existence and uniqueness theorem. Consider our usual initial value problem:

$$\begin{aligned} x' &= f(x) \\ x(0) &= x_0 \end{aligned} \tag{21.1}$$

where $f\colon E \to \mathbb{R}^n$ is continuously differentiable on the open subset $E \subset \mathbb{R}^n$ and $x_0 \in E$. The first refinement (dependence on parameters) says that if we deform f smoothly and move x_0 slightly, then the solution deforms smoothly. The second refinement says that the solution $x(t)$ to our initial value problem exists on a uniquely determined maximal interval about $t = 0$.

Theorem. (Dependence on parameters.) Let E be an open subset of $\mathbb{R}^{n+m}$ containing the point (x_0, μ_0) where $x_0 \in \mathbb{R}^n$ and $\mu_0 \in \mathbb{R}^m$, and assume $f \in C^1(E)$. Then there is a neighborhood[1] $N(x_0) \subseteq \mathbb{R}^n$ of x_0, a neighborhood $N(\mu_0) \subseteq \mathbb{R}^m$ of μ_0, and an $a > 0$ such that for all $y \in N(x_0)$ and for all $\mu \in N(\mu_0)$, the initial value problem

$$\begin{aligned} x' &= f(x, \mu) \\ x(0) &= y \end{aligned}$$

has a unique solution $x = x(t, y, \mu)$ with $x \in C^1(R)$ where $R := [-a, a] \times N(x_0) \times N(\mu_0)$.

Example. Let $A \in M_n(\mathbb{R})$ and $x_0 \in \mathbb{R}^n$. Then the solution to the system $x' = Ax$ with $x(0) = x_0$ is $x(t, x_0, A) = e^{At}x_0$, which is a smooth function of t, A, and x_0. In this case, $m = \binom{n}{2}$, and we identify a point $\mu \in \mathbb{R}^m$ with a matrix A_μ whose entries, read from left-to-right, top-to-bottom form μ. Thus, $f(x, \mu) = A_\mu x$.

[1] A *neighborhood* of a point is any set that contains an open set containing the point.

Theorem. Consider our initial value problem with $f \in C^1(E)$ and initial condition x_0. There is an interval $J = (\alpha, \beta)$ with $\alpha, \beta \in \mathbb{R} \cup \{\pm\infty\}$ and a solution $x(t)$ defined for $t \in J$ such that if $y(t)$ is any other solution defined on an interval I, then $I \subseteq J$ and $x(t) = y(t)$ on I. Further, if $\beta \in \mathbb{R}$, i.e., if $\beta \neq \infty$, then given any compact (closed and bounded) subset $K \subset E$, then there exists $t \in J$ such that $x(t) \notin K$.

The interval J is called the *maximal interval of existence* and is clearly uniquely determined.

Week 8, Monday: Stable manifold theorem

Last lecture, we started investigating the effect of replacing $f(x)$ with Df_{x_0} in (1) at an equilibrium point x_0, i.e., at a point where $f(x_0) = 0$. The first theorem we'll consider which makes this comparison precise is the *stable manifold theorem.* To state the theorem we need to formally introduce the *flow* of a vector field, and the idea of a *manifold.*

Flow. For each $x_0 \in E$, let $I(x_0)$ be the maximal interval of existence of the solution to (1) with initial condition x_0. Then let

$$\Omega := \{(t, x_0) \in \mathbb{R} \times E : t \in I(x_0)\}.$$

For each $(t, x_0) \in \Omega$, let $\phi(t, x_0)$ be the solution to (1) with initial condition x_0 evaluated at time $t \in I(x_0)$. This defines a mapping

$$\phi \colon \Omega \to \mathbb{R}^n$$

called the *flow* of the vector field $f \colon E \to \mathbb{R}^n$. For each $t \in I(x_0)$ we define

$$\phi_t(x_0) := \phi(t, x_0).$$

Our text (Section 2.5) establishes the following properties for the flow:

1. $\phi_0(x_0) = x_0$
2. $\phi_s(\phi_t(x_0)) = \phi_{s+t}(x_0)$
3. $\phi_{-t}(\phi_t(x_0)) = x_0$

wherever these expressions make sense.

Example. Consider the case of a linear system, in which $f(x) = Ax$ for some $A \in M_n(\mathbb{R})$. Here $E = \mathbb{R}^n$, and for each $x_0 \in \mathbb{R}^n$, the solution is

$$\phi_t(x_0) = x(t) = e^{At} x_0,$$

and the maximal interval of existence is $I(x_0) = \mathbb{R}$. So $\Omega = \mathbb{R}^{n+1}$, and the above properties for the flow are easily verified in this special case. For instance,

$$\phi_s(\phi_t(x_0)) = e^{As}(e^{At}x_0) = e^{A(s+t)}x_0 = \phi_{s+t}(x_0).$$

Manifolds. Roughly speaking, a manifold is a object that can be constructed from a collection of open subsets of $\mathbb{R}^n$ and a set of instructions for gluing these open sets together. A quintessential example is given by an ordinary world atlas. Each page consists of a flattened out map of a piece of the earth. There will be pairs of pages that overlap along boundaries representing the same regions. The drawings of features of the earth on these pages implicitly provide instructions for gluing the pages together. If the pages where made of moldable putty, then it would be possible to piece these pages together to make a shape. One possible result, among others would be a sphere, and so we say the sphere is a manifold. It is two-dimensional since we glue together open subsets of $\mathbb{R}^2$ to make it. We now move on to the formal definition.

Definition. A *metric space* is a set X with a *distance function* or *metric*,

$$d\colon X \times X \to \mathbb{R}$$

that is positive definite, symmetric, and obeys the triangle inequality:

1. $d(x, y) \geq 0$ with $d(x, y) = 0$ if and only if $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, y) \leq d(x, z) + d(z, y)$.

Every metric space (X, d) is a topological space where a subset $U \subseteq X$ is *open* if for each $u \in U$, there exists $r > 0$ such that the open ball of radius r centered at u is contained in U:

$$B(u, r) := \{x \in X : d(u, x) < r\} \subseteq U.$$

Definition. Two subsets A, B of a metric space X are *homeomorphic* if there exists a continuous bijection $f\colon A \to B$ with continuous inverse. The mapping f is then called a *homeomorphism* from A to B. (More generally, two topological spaces U, V are homeomorphic if there is a continuous bijection $f\colon U \to V$ with continuous inverse.)

Definition. An n-dimensional differentiable manifold is a connected metric space[1] M and an open covering $\{U_\alpha\}$ (so for each α in some index set, U_α is an open subset of M and $M = \cup_\alpha U_\alpha$) such that:

1. for all α, there is a homeomorphism

$$h_\alpha \colon U_\alpha \to V_\alpha$$

   where V_α is an open subset of $\mathbb{R}^n$, and

2. if $U_\alpha \cap U_\beta \neq \emptyset$, the mapping

$$h_\beta \circ h_\alpha^{-1} \colon h_\alpha(U_\alpha \cap U_\beta) \to h_\beta(U_\alpha \cap U_\beta)$$

   is continuously differentiable.

Each pair (h_α, U_α) is called a *chart*, and the collection of charts is called an *atlas*. The mapping $h_\beta \cap h_\alpha^{-1}$ are *transition functions*.

To go back to the rough description we made earlier: each chart (h_α, U_α) represents a page $h_\alpha(U_\alpha)$ in the atlas. The set U_α is a piece of the manifold (earth), and the mapping h_α is the rendering of that piece of the earth onto a flat piece of paper. On overlaps $U_\alpha \cap U_\beta$ on the manifold the corresponding pages of the atlas have overlaps $h_\alpha(U_\alpha \cap U_\beta)$ and $h_\beta(U_\alpha \cap U_\beta)$. We can glue these together with the transition function $h_\beta \circ h_\alpha^{-1}$.

Theorem. (Stable manifold theorem.) Let $E \subseteq \mathbb{R}^n$ and let $f \in C^1(E)$. Suppose that $f(0) = 0$ and that Df_0 has k eigenvalues with negative real part and $n - k$ eigenvalues with positive real part. Then there exists a k-dimensional differentiable manifold S tangent to the stable subspace E^s of the linearized system $x' = Df_0(x)$ at 0 and there exists an $(n-k)$-dimensional differentiable manifold U tangent to the unstable space E^u of the linearized system. Further

$$\lim_{t\to\infty} \phi_t(x_0) = 0$$

for any $x_0 \in S$ and

$$\lim_{t\to-\infty} \phi(x_0) = 0$$

for any $x_0 \in U$.

[1]More generally, M could be a second-countable Hausdorff toplogical space.

Week 8, Wednesday: Stable manifold theorem

STABLE MANIFOLD THEOREM

Review of stable, unstable, and center subspaces. Consider the linear system $x' = Ax$ for some $A \in M_n(\mathbb{R})$. Suppose that the generalized eigenvectors and their corresponding eigenvalues for A are $u_j + iv_j$ and $\lambda_j = a_j + ib_j$, respectively, for $j = 1, \ldots, n$. Thus, putting these vectors as columns in a matrix P, we have $P^{-1}AP = J$ where J is the Jordan form of A. The generalized eigenvectors $u_j + iv_j$ for which $b_j \neq 0$ come in conjugate pairs since A is a real matrix. Then the **stable, unstable, and center subspaces** for the system are, respectively,

$$\begin{aligned} E^s &:= \operatorname{Span}\{u_j, v_j : a_j < 0\} \\ E^u &:= \operatorname{Span}\{u_j, v_j : a_j > 0\} \\ E^c &:= \operatorname{Span}\{u_j, v_j : a_j = 0\}. \end{aligned}$$

Recall that up to a change of coordinates, the solution to the system is e^{Jt} and that for a Jordan block corresponding to $\lambda_j = a_j + ib_j$, we can factor out $e^{\lambda_j t} = e^{a_j t}(\cos(b_j t) + i\sin(b_j t))$, leaving a matrix that is polynomial in t:

$$e^{J_\ell(\lambda_j)t} = e^{\lambda_j t}\begin{pmatrix} 1 & t & \frac{t^2}{2!} & \ldots & \ldots & \frac{t^{\ell-1}}{(\ell-1)!} \\ 0 & 1 & t & \ldots & \ldots & \frac{t^{\ell-2}}{(\ell-2)!} \\ 0 & 0 & 1 & \ldots & \ldots & \frac{t^{\ell-3}}{(\ell-3)!} \\ & \ddots & & \vdots & \vdots & \vdots \\ 0 & \ldots & \ldots & 0 & 1 & t \\ 0 & \ldots & \ldots & \ldots & 0 & 1 \end{pmatrix}.$$

Thus, it is the signs of the a_j that determine the long-term behavior of the system.

Theorem. (Stable manifold theorem.) Let $E \subseteq \mathbb{R}^n$ and let $f \in C^1(E)$. Suppose that $f(0) = 0$ and that Df_0 has k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part. Let ϕ be the flow for the system $x' = f(x)$.

Then there exists a k-dimensional differentiable manifold S tangent to the stable subspace E^s of the linearized system $x' = Df_0(x)$ at 0 and there exists an $(n-k)$-dimensional differentiable manifold U tangent to the unstable space E^u of the linearized system. Further

$$\lim_{t\to\infty} \phi_t(p) = 0$$

for any $p \in S$ and

$$\lim_{t\to-\infty} \phi(p) = 0$$

for any $p \in U$.

Remark. To apply this theorem to an arbitrary equilibrium point x_0, make the change of coordinates $x \mapsto x - x_0$, find the stable and unstable manifolds at the origin, and translate back $x \mapsto x + x_0$.

Example. The system

$$\begin{aligned} x' &= -x - y^2 \\ y' &= y + x^2 \end{aligned}$$

has an equilibrium point at the origin. The Jacobian for $f(x,y) = (-x - y^2, y + x^2)$ is

$$Jf(x,y) = \begin{pmatrix} -1 & -2y \\ 2x & 1 \end{pmatrix}.$$

Therefore,

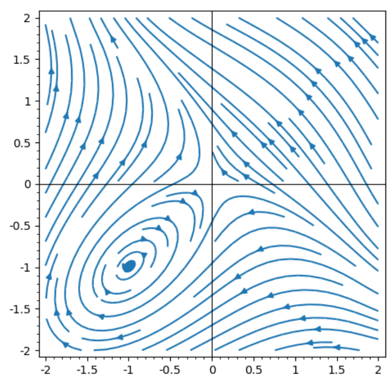
$$Jf(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and the linearized system is

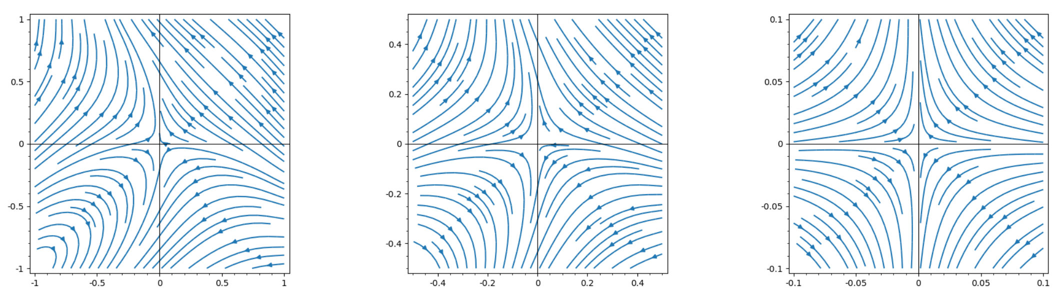
$$\begin{aligned} x' &= -x \\ y' &= y. \end{aligned}$$

(The linearized system in this case is easy to read off of the original system in this case since the equilibrium point is the origin and f has components that are polynomials since f is it's own Taylor expansion at the origin.)

The main thing that concerns us, though, is that the eigenvalues for $Df_{(0,0)}$ are ± 1. The eigenspace for -1 is spanned by $(1,0)$, i.e., the x-axis, and the eigenspace for 1 is spanned by $(0,1)$, the y-axis. So a stable manifold for our original system should be tangent to the x-axis and an unstable manifold should be tangent to the y-axis at the origin. Here is a picture of the flow of the vector field f:



From the picture, we can see that the vector field is not tangent to the stable and unstable spaces for the linearized system at the origin everywhere. The stable manifold theorem is a statement about what is happening locally, very close to the equilibrium point. Below, we zoom in on the origin:



Sketch of proof of the stable manifold theorem. The proof of the stable manifold theorem, like the proof of the fundamental existence and uniqueness theorem can be done by the method of successive approximations.

We start with some "pre-processing": As mentioned above, if the equilibrium point x_0 is not the origin, first replace x by $x - x_0$. Suppose that has been done. Second, write

$$x' = f(x) = Jf(0)x + (f(x) - Jf(0)x)\,.$$

Defining $F(x) := f(x) - Jf(0)x$, our system becomes

$$.$$

Third, choose an $n \times n$ real matrix P such that

$$P^{-1}Jf(0)P = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where A has k eigenvalues with negative real parts and B has $n-k$ eigenvalues with positive real parts. Finally, make the change of variables $y = P^{-1}x$. Then

$$\begin{aligned} x' = Jf(0)x + F(x) \quad &\Rightarrow \quad Py' = Jf(0)Py + F(Py) \\ &\Rightarrow \quad y' = P^{-1}Jf(0)Py + P^{-1}F(Py). \end{aligned}$$

Define $G(y) = P^{-1}F(Py)$ to get the system

$$y' = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} y + G(y). \tag{23.1}$$

Each step we've made is reversible. So solving this system is equivalent to solving the original system.

We now find stable and unstable manifolds through the method of successive approximations. Define

$$U(t) := \begin{pmatrix} e^{At} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad V(t) := \begin{pmatrix} 0 & 0 \\ 0 & e^{Bt} \end{pmatrix}$$

so that

$$e^{\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)t} = U(t) + V(t).$$

For $t \in \mathbb{R}$ and $a \in \mathbb{R}^n$, define an operator T on $\mathbb{R}^n$-valued functions u with domain in a region near the origin in $\mathbb{R} \times \mathbb{R}^n$ by

$$(Tu)(t,a) := U(t)a + \int_{s=0}^{t} U(t-s)G(u(s,a))\,ds - \int_{s=t}^{\infty} V(t-s)G(u(s,a))\,ds. \tag{23.2}$$

Now use the method of successive approximations starting with

$$u^{(0)}(t,a) = 0 \in \mathbb{R}^n.$$

Calculations like those we did for the proof of the fundamental existence and uniqueness theorem show the approximations $u^{(m)}(t,a)$ converge to a fixed point $u(t)$ of T for t in a small interval about the origin and for a restricted to a sufficiently small neighborhood of the origin in $\mathbb{R}^n$.

A stable manifold for equation (23.1) is given as the set of points

$$(a_1, \ldots, a_k, u_{k+1}(0, a_1, \ldots, a_k, 0, \ldots, 0), \ldots, u_n(0, a_1, \ldots, a_k, 0, \ldots, 0))$$

as $(a_1, \ldots, a_k)$ varies in a neighborhood of the origin in $\mathbb{R}^k$. We get a stable manifold for the original system by applying P to these points, since $y = P^{-1}x$, then translating back $x \mapsto x + x_0$, if the original equilibrium point was not the origin.

To find an unstable manifold, replace t by $-t$ to get the system

$$y' = -\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} y - G(y),$$

since $(y(-t))' = -y'(-t)$. However, now note that $-A$ has k positive eigenvalues and $-B$ has $n-k$ negative eigenvalues, so to apply the above argument, we need to swap coordinates $\phi\colon y \mapsto (y_{k+1}, \ldots, y_n, y_1, \ldots, y_k)$ to get the system

$$(\phi(y))' = \begin{pmatrix} -B & 0 \\ 0 & -A \end{pmatrix} \phi(y) - G(\phi(y)),$$

apply the method of successive approximations, then swap back by applying ϕ^{-1} to the points in the resulting manifold. □

As evidence for the reasonableness of the method presented in the sketch above suppose that $y(t)$ is a solution to equation (23.2) with (i) initial condition $y(0)$ close to 0 and such that (ii) $y(t)$ is bounded as $t \to \infty$. We will show that y must satisfy equation (23.2) (and thus will be a fixed point of the iterative process). Let

$$M := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

so that the system becomes

$$y' = My + G(y).$$

Then

$$\begin{aligned}
y' = My + G(y) \quad &\Rightarrow \quad e^{-Mt}y' = e^{-Mt}My + e^{-Mt}G(y) \\
&\Rightarrow \quad e^{-Mt}y' = Me^{-Mt}y + e^{-Mt}G(y) \\
&\Rightarrow \quad e^{-Mt}y' - Me^{-Mt}y = e^{-Mt}G(y) \\
&\Rightarrow \quad (e^{-Mt}y)' = e^{-Mt}G(y) \\
&\Rightarrow \quad \int_{s=0}^{t} (e^{-Ms}y(s))'\,ds = \int_{s=0}^{t} e^{-Ms}G(y(s)\,ds \\
&\Rightarrow \quad e^{-Mt}y(t) - y(0) = \int_{s=0}^{t} e^{-Ms}G(y(s))\,ds \\
&\Rightarrow \quad y(t) - e^{Mt}y(0) = \int_{s=0}^{t} e^{M(t-s)}G(y(s))\,ds
\end{aligned}$$

$$
\begin{aligned}
\Rightarrow \quad y(t) &= e^{Mt}y(0) + \int_{s=0}^{t} e^{M(t-s)}G(y(s))\,ds \\
\Rightarrow \quad y(t) &= (U(t)+V(t))y(0) \\
&\qquad + \int_{s=0}^{t} (U(t-s)+V(t-s))G(y(s))\,ds \\
\Rightarrow \quad y(t) &= (U(t)+V(t))y(0) + \int_{s=0}^{t} U(t-s)G(y(s))\,ds \\
&\qquad + \int_{s=0}^{\infty} V(t-s))G(y(s))\,ds - \int_{s=t}^{\infty} V(t-s))G(y(s))\,ds
\end{aligned}
$$

To see that the integrals here are all bounded, first note that since y is bounded as $t \to \infty$ (by assumption) and G is continuous, we have that $G(y(s))$ is bounded as $s \to \infty$. Next note that since the real part of the eigenvalues of B are positive, $V(t-s)$ is bounded as $s \to \infty$ (recall that $V(t-s) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & e^{B(t-s)} \end{smallmatrix}\right)$). Continuing,

$$
\begin{aligned}
y(t) &= (U(t)+V(t))y(0) + \int_{s=0}^{t} U(t-s)G(y(s))\,ds \\
&\qquad + \int_{s=0}^{\infty} V(t-s))G(y(s))\,ds - \int_{s=t}^{\infty} V(t-s))G(y(s))\,ds \\
\Rightarrow y(t) &= U(t)y(0) + V(t)\left(y(0) + \int_{s=0}^{\infty} V(-s)G(y(s))\,ds\right) \qquad (\star) \\
&\qquad + \int_{s=0}^{t} U(t-s)G(y(s))\,ds - \int_{s=t}^{\infty} V(t-s))G(y(s))\,ds
\end{aligned}
$$

Consider the above equation. On the left, we have $y(t)$, which is bounded as $t \to \infty$. One the right, considering the eigenvalues of A and B we see that first, third, and fourth summands are bounded as $t \to \infty$. This implies that

$$
V(t)\left(y(0) + \int_{s=0}^{\infty} V(-s)G(y(s))\,ds\right)
$$

is bounded as $t \to \infty$. But recall that

$$
V(t) = \begin{pmatrix} 0 & 0 \\ 0 & e^{Bt} \end{pmatrix}
$$

where B has $n-k$ eigenvalues, each with positive real parts. Since

$$y(0) + \int_{s=0}^{\infty} V(-s)G(y(s))\, ds$$

is bounded (in fact, constant), this means that

$$V(t)\left(y(0) + \int_{s=0}^{\infty} V(-s)G(y(s))\, ds\right) = 0$$

(Note that the above equation is a product of two matrices. So we cannot conclude that either of the factors is the zero matrix.) From equation ($\star$), above, it follows that

$$y(t) = U(t)y(0) + \int_{s=0}^{t} U(t-s)G(y(s))\, ds - \int_{s=t}^{\infty} V(t-s))G(y(s))\, ds,$$

as we wanted to show.

Week 8, Friday: Hartman-Grobman theorem

GLOBAL STABLE AND UNSTABLE MANIFOLDS

Let E be an open subset of $\mathbb{R}^n$ containing the origin, 0, and let $f\colon E \to \mathbb{R}^n$ be continuously differentiable. Consider the system of differential equations $x' = f(x)$. Suppose 0 is an equilibrium point and that Df_0 has k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part. By the stable manifold theorem, in a neighborhood of 0 there exists a k-dimensional stable manifold S and an $n-k$-dimensional unstable manifold U. The manifold S is tangent at 0 to the stable space E^s for the linearized system $x' = Df_0(x)$. Similarly, U is tangent at 0 to the unstable space E^u for the linearized system. Further, if $\phi_t(x)$ is the flow for the system, then

$$\lim_{t\to\infty} \phi_t(p) = 0$$

for all $p \in S$ and

$$\lim_{t\to-\infty} \phi_t(p) = 0$$

for all $p \in U$.

Define the *global stable and unstable manifolds* at the equilibrium point 0 by

$$W^s(0) := \cup_{t\leq 0}\phi_t(S)$$

and

$$W^u(0) := \cup_{t\geq 0}\phi_t(U),$$

respectively. Here, for any subset $X \subset E$,

$$\phi_t(X) := \{\phi_t(x)\colon x \in X\}.$$

It turns out that these manifolds (i) do not depend on our choice of local stable and unstable manifolds S and U, (ii) are invariant under ϕ_t, and (iii) for all $p \in W^s(0)$,

$$\lim_{t\to\infty} \phi_t(p) = 0$$

and for all $p \in W^u(0)$,

$$\lim_{t \to -\infty} \phi_t(p) = 0.$$

Remark. There is also version of the stable manifold theorem that applies to equilibrium points where the linearization has eigenvalues with real part equal to zero. It states that if the linearization has k eigenvalues with positive real part, j eigenvalues with negative real part, and $m = n - k - j$ eigenvalues with zero real part, then there are manifolds W^s, W^u, and W^c tangent to stable, unstable, and central spaces, respectively, of the linearization having dimensions k, j, and m, respectively. These spaces are invariant under the flow of the system. See our text, Section 2.7.

Hartman-Grobman theorem

We again consider a system $x' = f(x)$ as above with equilibrium point $x_0 = 0$. (For an arbitrary equilibrium point x_0, just replace x by $x - x_0$.) We again assume the linearized system has no eigenvalues with real part equal to 0. These equilibrium points are called *hyperbolic equilibrium points*. Roughly, the Hartman-Grobman theorem says that in a neighborhood of x_0, the system $x' = f(x)$ and the linearized system $x' = Df_{x_0}(x)$ are qualitatively the same, in a way to be made precise below.

Theorem. (Hartman-Grobman) Let E be an open subset of $\mathbb{R}^n$ containing the origin, and let $f \colon E \to \mathbb{R}^n$ be continuously differentiable with Jacobian matrix Jf. Suppose that 0 is a hyperbolic equilibrium point of the system $x' = f(x)$. Then there exist open neighborhoods U and V of the origin and a homeomorphism (i.e., a continuous bijection with continuous inverse)

$$H \colon U \to V$$

with $H(0) = 0$ having the following property: for all $x_0 \in U$, there is an interval $I \subseteq \mathbb{R}$ containing the origin such that for all $t \in I$,

$$H(\phi_t(x_0)) = e^{Jf(0)t} H(x_0).$$

The theorem says H maps trajectories of the system $x' = f(x)$ to trajectories of the linearized system $x' = Jf(0)x$ in a neighborhood of the origin. (Nonzero equilibria are handled by translating to the origin, as usual.) The proof of the theorem is outlined in Section 2.8 of our text and goes, again, by the method of successive approximations.

Example. Consider the system

$$\begin{aligned} x' &= -x \\ y' &= y + x^2. \end{aligned}$$

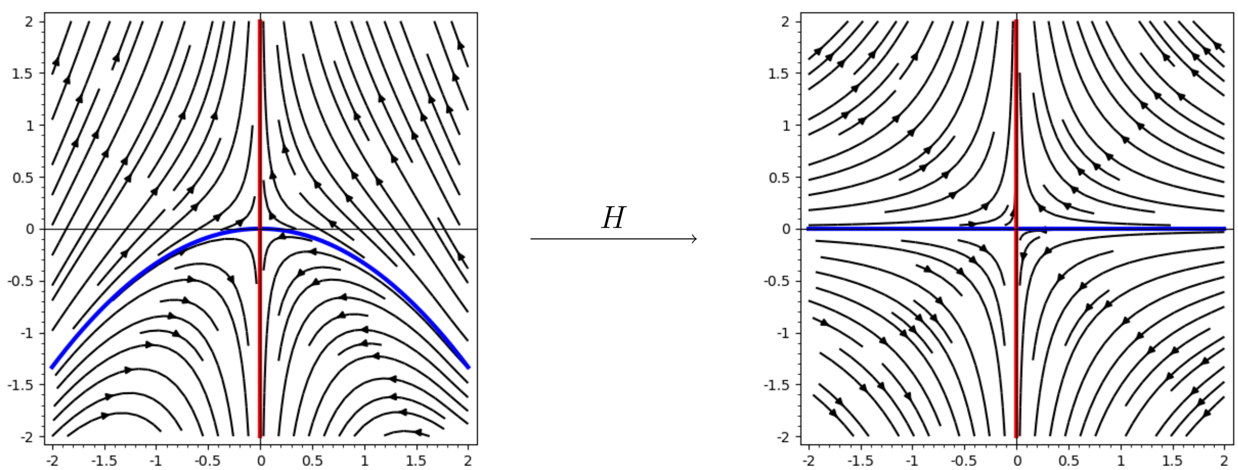
The origin is a hyperbolic equilibrium point, and the linearized system there is

$$\begin{aligned} x' &= -x \\ y' &= y. \end{aligned}$$

Our text shows how to apply the method of successive approximations to find the homeomorphism

$$H(x, y) = \left(x, y + \frac{1}{3}x^2 \right).$$

The effect of the mapping is illustrated below with the stable manifolds in blue and the unstable manifolds in red:



The nonlinear system can be solved using the methods we covered during week five of the semester, and the solution with initial condition (x_0, y_0) is

$$\begin{aligned} x(t) &= x_0 e^{-t} \\ y(t) &= \left(y_0 + \frac{1}{3}x_0^2 \right) e^t - \frac{1}{3}x_0^2 e^{-2t}. \end{aligned}$$

To find the stable manifold, we find the points (x_0, y_0) such that the solution with that initial points converges to $(0,0)$ as $t \to \infty$. For the unstable manifold, we do the same but with $t \to -\infty$. We find

$$W^s(0,0) = \left\{ \left(x, -\frac{1}{3}x^2 \right) : x \in \mathbb{R} \right\}$$
$$W^u(0,0) = \{(0,y) : y \in \mathbb{R}\}.$$

The solution to the linearized system is

$$x(t) = x_0 e^{-t}$$
$$y(t) = y_0 e^{t}$$

with stable and unstable spaces

$$E^s = \{(x,0) : x \in \mathbb{R}\}$$
$$E^u = \{(0,y) : y \in \mathbb{R}\}.$$

Applying H to the solution of the nonlinear system gives

$$\begin{aligned} H(\phi_t(x,y)) &= H\left(xe^{-t}, \left(y + \frac{1}{3}x^2 \right) e^t - \frac{1}{3}x^2 e^{-2t} \right) \\ &= \left(xe^{-t}, \left(y + \frac{1}{3}x^2 \right) e^t \right) \\ &= \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \begin{pmatrix} xe^{-t} \\ \left(y + \frac{1}{3}x^2 \right) e^t \end{pmatrix} \\ &= e^{Jf(0,0)t} H(x,y). \end{aligned}$$

The stable and unstable manifolds for the nonlinear system are mapped by H to the stable and unstable spaces, respectively, for the linear system:

$$H\left(x, -\frac{1}{3}x^2 \right) = (x, 0),$$

and

$$H(0,y) = (0,y).$$

Week 9, Monday: Stability and Liapunov functions

Liapunov functions and stability

Definition. An equilibrium point x_0 for a system $x' = f(x)$ is *stable* if for each open neighborhood U of x_0, there exists another open neighborhood W of x_0 such that if $p \in W$, then $\phi(t,p) \in U$ for all $t \geq 0$. Otherwise, x_0 is *unstable.* We say x_0 is *asymptotically stable* if it has an open neighborhood W such that

$$\lim_{t\to\infty} \phi_t(p) = x_0$$

for all $p \in W$.

Facts.

1. Surprisingly, an equilibrium point can be both unstable and asymptotically stable! We'll see an example in the homework.

2. Suppose x_0 is a hyperbolic equilibrium point, i.e., it's linearized system has no eigenvalues with real part equal to 0. To analyze the stability of x_0, we use Hartman-Grobman to replace the system $x' = f(x)$ with its linearization $x' = Df_{x_0}(x)$ at x_0. If all eigenvalues of Df_{x_0} have negative real part, then x_0 is stable and asymptotically stable, and the approach of a trajectory to x_0 is exponential in time. Otherwise, some eigenvalue has positive real part, and x_0 is unstable.

3. In any case, it turns out that if an equilibrium point x_0 is stable, then no eigenvalue of Df_{x_0} has positive real part (even in the non-hyperbolic case).

Liapunov functions. Let x_0 be an equilibrium point. Suppose there is a way to assign a smoothly changing "temperature" to each point in E such that: (i) the temperature at x_0 is 0, (ii) the temperature at every other point is positive. Could we determine stability only knowing the temperatures along trajectories? This is the idea behind the notion of a Liapunov function. (Below, we label the temperature function by V.)

Given $V\colon E \to \mathbb{R}$ and $p \in E$, we define

$$\dot{V}(p) = \frac{d}{dt} V(\phi_t(p)) \bigg|_{t=0}.$$

Thus $\dot{V}(p)$ tells us how fast the temperature is changing along the solution trajectory as it passes through p.

Theorem. Let $f \in C^1(E)$ and $f(x_0) = 0$. Let $V\colon E \to \mathbb{R}$ also be C^1 (continuously differentiable). Suppose that $V(p) \geq 0$ and $V(p) = 0$ if and only if $p = x_0$. Then:

1. If $\dot{V}$ is negative semidefinite ($\dot{V}(p) \leq 0$ for all $p \in E \setminus \{x_0\}$) then x_0 is stable.

2. If $\dot{V}$ is negative definite ($\dot{V}(p) < 0$ for all $p \in E \setminus \{x_0\}$) then x_0 is asymptotically stable.

3. If $\dot{V}$ is positive definite ($\dot{V}(p) > 0$ for all $p \in E \setminus \{x_0\}$), then x_0 is unstable.

Definition. A function satisfying the hypotheses of the previous theorem is called a *Liapunov function.*

Happily, thanks to the chain rule, the conditions on $\dot{V}$ in the theorem can be verified *without solving the system*:

Proposition. With V as above,

$$\dot{V}(p) = \nabla V(p) \cdot f(p).$$

Proof. Let $\psi(t) := \phi_t(p)$. Apply the chain rule:

$$\begin{aligned}
J(V \circ \psi)(0) &= JV(\psi(0)) J\psi(0) \\
&= JV(p) J\psi(0) \\
&= \begin{pmatrix} \frac{\partial V}{\partial x_1}(p) & \dots & \frac{\partial V}{\partial x_n}(p) \end{pmatrix} \begin{pmatrix} \psi_1'(0) \\ \vdots \\ \psi_n'(0) \end{pmatrix} \\
&= \nabla V(p) \cdot \psi'(0).
\end{aligned}$$

Now, ψ is the solution to the system $x' = f(x)$ with initial condition p. Therefore, $\psi'(t) = f(\psi(t))$, and $\psi'(0) = f(\psi(0)) = f(p)$. The result follows. $\square$

Example. Consider the system

$$\begin{aligned} x' &= -y^3 \\ y' &= x^3. \end{aligned}$$

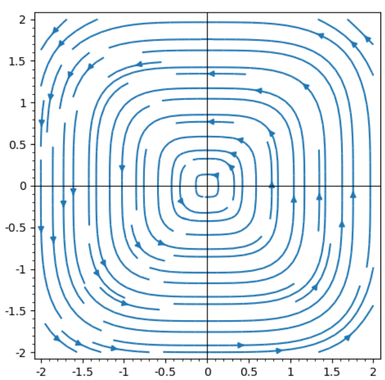
The origin is a non-hyperbolic equilibrium point and

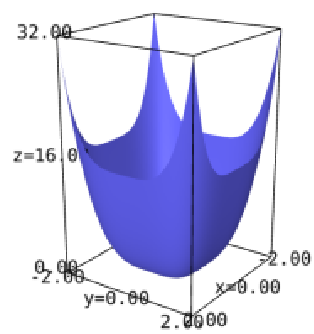
$$V(x, y) = x^4 + y^4$$

is a Liapunov function for that point. (Clearly, V is smooth and $V(x, y) \geq 0$ with equality only at the origin.) For any trajectory $(x, y) = (x(t), y(t))$, we have

$$\dot{V}(x, y) = 4x^3 x' + 4y^3 y' = 4x^3(-y^3) + 4y^3(x^3) = 0.$$

Hence, the origin is stable. In fact, our calculation shows that $V(\phi_t(p))$ is a constant as a function of t. In other words, trajectories (solutions) sit on level sets for V, as seen in the following:





Proof of theorem. We may assume $x_0 = 0 \in \mathbb{R}^n$ is the equilibrium point.
(1) Suppose that $\dot{V}(p) \leq 0$ for all $p \in E \setminus \{x_0\}$. Choose $\varepsilon > 0$ such that the open ball $B_\varepsilon(x_0)$ of radius ε centered at x_0 is contained in E. Let

$$\overline{B_\varepsilon(x_0)} := \{x \in \mathbb{R}^n : |x - x_0| \leq \varepsilon\}.$$

Replacing ε by $\varepsilon/2$, if necessary, we may assume $\overline{B_\varepsilon(x_0)} \subset E$. Let

$$\alpha := \min_{|x|=\varepsilon} V(x),$$

the minimum of V on the boundary of $\overline{B_\varepsilon(x_0)}$. The function V achieves its minimum on the boundary since V is continuous and the boundary is compact (closed and bounded). Since the minimum is achieved at some point on the boundary and V is strictly greater than 0 away from the origin, we have $\alpha > 0$.

Define
$$W := \{x \in B_\varepsilon(x_0) : V(x) < \alpha\}.$$
We think of W as the set of points in B_ε whose "temperature" is less that α, the minimum temperature on the boundary of B_ε. Then W is an open[1] neighborhood of the origin, and no solution starting at a point in W can leave W since V is nonincreasing on solution curves. Thus x_0 is stable.

(2) Suppose now that $\dot{V}(p) < 0$ for all $p \in E \setminus \{x_0\}$. As in the proof for part (1), we choose $\varepsilon > 0$ so that $\overline{B_\varepsilon(x_0)} \subset E$. We let
$$\alpha := \min_{|x|=\varepsilon} V(x),$$
and take
$$W := \{x \in B_\varepsilon(x_0) : V(x) < \alpha\}.$$
Since $\dot{V}(p) < 0$ for all $p \in E \setminus \{x_0\}$, we saw in the proof of part (1) that solution trajectories starting in W never leave W. We would like to show that $\lim_{t\to\infty} \phi_t(p) = 0$ for all $p \in W$. Pick any sequence $t_1 < t_2 < \dots$ such that $t_n \to \infty$, and consider the sequence
$$\{\phi(t_n, p)\}.$$
By part (1), this sequence never leaves W, and hence it is contained in the closure $\overline{W} \subseteq \overline{B_\varepsilon(x_0)}$, which is compact. So by the Bolzano-Weierstrass theorem, there exists a convergent subsequence. This means that there is a subsequence t_{n_k} such that
$$\lim_{k\to\infty} \phi(t_{n_k}, p) = q$$
for some $q \in \overline{W}$. For ease of writing, replace our original sequence with the subsequence $\{t_{n_k}\}_k$. We then have
$$\lim_{n\to\infty} \phi(t_n, p) = q.$$
We would like to show that $q = x_0 = 0$, and we will do this by contradiction. Suppose that $q \neq 0$. Then $V(q) > 0$. Also since V is strictly decreasing along trajectories, we have
$$V(q) > V(\phi(1, q)).$$
Since $\lim_{n\to\infty} \phi(t_n, p) = q$, by continuity of solutions with respect to both time and initial conditions, and by continuity of V, there exists an integer N large enough so

[1] The set W is open since $W = V^{-1}((-\infty, \alpha))$, and by definition of continuity, the inverse image of an open subset under a continuous function is continuous.

that $\phi(t_N, p)$ is close enough to q so that $V(\phi(1, \phi(t_N, p)))$ is close enough to $V(\phi(1, q))$ so that

$$V(\phi(1 + t_N, p)) = V(\phi(1, \phi(t_N, p)) < V(q).$$

Since $t_n \to \infty$, we can find M such that $t_M > 1 + t_N$. Then, since V is strictly decreasing along trajectories, we have

$$V(q) > V(\phi(1 + t_N, p)) > V(\phi(t_M, p))$$

This is a problem: since V stricty decreases along trajectories and V is continuous, we have that the sequence $\{V(\phi(t_n, p)\}$ is strictly decreasing and converges to $V(q)$. So in contradiction to the inequalities displayed above,

$$V(\phi(t_M, p) > V(q).$$

We have shown that $q = 0$ and that there is a sequence $\{t_n\}$ such that $\lim_{n\to\infty} \phi(t_n, p) = q = 0$. We now need to show $\lim_{t\to\infty} \phi(t, p) = x_0 = 0$. If not, there exists an $\eta > 0$ such that for all n, there exists $s_n > n$ such that

$$|\phi(s_n, p)| \geq \eta > 0. \tag{25.1}$$

We may assume that the sequence s_n is increasing. However, by Bolzano-Weierstrass, there again exists a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ such that $\phi(s_{n_k}, p)$ converges, and as we have seen, it must converge to 0. But that's impossible in light of (25.1).

(3) Finally, now suppose that $\dot{V}(p) > 0$ for all $p \in E \setminus \{x_0\}$. Choose $\varepsilon > 0$ such that $\overline{B_\varepsilon(0)} \subset E$. We'll show that given any point $p \in E$, we have that $\phi_t(p)$ leaves $B_\varepsilon(0)$ at some point, i.e., there exists $t \geq 0$ such that $|\phi_t(p)| > \varepsilon$. Hence, x_0 is unstable.

Given $p \in E \setminus \{0\}$, since V is strictly increasing on trajectories,

$$V(\phi_t(p)) > V(\phi_0(p)) = V(p) > 0$$

for all $t > 0$. Thus, $\phi_t(p)$ is bounded away from 0. Say $|\phi_t(p)| \geq \eta > 0$ for all $t \geq 0$. If $\eta \geq \varepsilon$, then we are done since $|p| = |\phi_0(p)| \geq \eta > \varepsilon$, which says p is already out of $B_\varepsilon(x_0)$. Otherwise, define

$$m := \min_{y:\eta\leq|y|\leq\varepsilon} \dot{V}(y),$$

which exists since $\dot{V}$ is continuous and y is restricted to a compact set. In fact, for that same reason, $m = \dot{V}(q)$ for some point in the set over which we are minimizing. Therefore, $m > 0$. Supposing for contradiction that $\phi_t(p)$ stays inside $B_\varepsilon(x_0)$ for all $t \geq 0$, we have $\dot{V}(\phi_t(p)) \geq m$ for all $t \geq 0$. Hence,

$$V(\phi_t(p)) - V(p) = V(\phi_t(p)) - V(\phi_0(p)) = \int_{s=0}^{t} \dot{V}(\phi_s(p))\, ds \geq mt \to \infty$$

as $t \to \infty$. But since V is continuous, it achieves a maximum on $\overline{B_\varepsilon(x_0)}$—a contradiction. $\square$

Example. Consider the system

$$\begin{aligned} x' &= -2y + yz \\ y' &= x - xz \\ z' &= xy. \end{aligned}$$

The Jacobian at the origin is

$$J(0) = \begin{pmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The characteristic polynomial is

$$\det \begin{pmatrix} -x & -2 & 0 \\ 1 & -x & 0 \\ 0 & 0 & -x \end{pmatrix} = -x^3 - 2x = -x(x^2 + 2).$$

So the eigenvalues are $0, \pm\sqrt{2}i$. So the origin is a nonhyperbolic equilibrium point. To determine stability, we look for a suitable Liapunov function. We guess a function of the form

$$V = ax^2 + by^2 + cz^2$$

with positive constants a, b, c. We have

$$\begin{aligned} \dot{V} &= 2axx' + 2byy' + 2czz' \\ &= 2ax(-2y + yz) + 2by(x - xz) + 2cz(xy) \\ &= 2(-2a + b)xy + 2(a - b + c)xyz. \end{aligned}$$

Take $a = c = 1$ and $b = 2$, and we get $V = x^2 + 2y^2 + z^2$ with $\dot{V} = 0$. This means that trajectories stay on the ellipsoids that are level sets of V. $\square$

Week 9, Friday: Liapunov functions

LIAPUNOV FUNCTIONS

Theorem. Let $f \in C^1(E)$ and $f(x_0) = 0$. Let $V: E \to \mathbb{R}$ also be C^1 (continuously differentiable). Suppose that $V(p) \geq 0$ and $V(p) = 0$ if and only if $p = x_0$. Then:

1. If $\dot{V}$ is negative semidefinite ($\dot{V}(p) \leq 0$ for all $p \in E \setminus \{x_0\}$) then x_0 is stable.
2. If $\dot{V}$ is negative definite ($\dot{V}(p) < 0$ for all $p \in E \setminus \{x_0\}$) then x_0 is asymptotically stable.
3. If $\dot{V}$ is positive definite ($\dot{V}(p) > 0$ for all $p \in E \setminus \{x_0\}$), then x_0 is unstable.

Proof. As before, we may assume $x_0 = (0, 0)$ is the equilibrium point. Part (1) was proved in the last lecture.

(2) Last time, we were in the midst of proving part (2). Using the notation from last time, so far, we have shown that for every sequence $t_1 < t_2 < \cdots$ such that $\lim_{n\to\infty} t_n = \infty$, there exists a subsequence $\{t_{n_k}\}$ such that $\lim_{k\to\infty} \phi(t_{n_k}, p) = 0 \in \mathbb{R}^n$.

We now need to show $\lim_{t\to\infty} \phi(t, p) = x_0 = 0$. If not, there exists an $\eta > 0$ such that for all n, there exists $t_n > n$ such that

$$|\phi(t_n, p)| \geq \eta > 0. \tag{26.1}$$

We may assume that the sequence t_n is increasing. However, by Bolzano-Weierstrass, there again exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that $\phi(t_{n_k}, p)$ converges, and as we have seen, it must converge to 0. But that's impossible in light of (26.1).

(3) Finally, now suppose that $\dot{V}(p) > 0$ for all $p \in E \setminus \{x_0\}$. Choose $\varepsilon > 0$ such that $\overline{B_\varepsilon(0)} \subset E$. We'll show that given any point $p \in B_\varepsilon(0) \setminus \{0\}$, we have that $\phi_t(p)$ leaves $B_\varepsilon(0)$ at some point, i.e., there exists $t \geq 0$ such that $|\phi_t(p)| \geq \varepsilon$.

Given $p \in B_\varepsilon(0) \setminus \{0\}$, since V is strictly increasing on trajectories,

$$V(\phi_t(p)) > V(\phi_0(p)) = V(p) > 0$$

for all $t > 0$. Thus, $\phi_t(p)$ is bounded away from 0. Say $|\phi_t(p)| \geq \eta > 0$ for all $t \geq 0$. Since $\eta \leq |\phi_0(p)| = |p| < \varepsilon$, it follows that $\eta < \varepsilon$. Define

$$m := \min_{y: \eta \leq |y| \leq \varepsilon} \dot{V}(y),$$

which exists since $\dot{V}$ is continuous and y is restricted to a compact set. In fact, for that same reason, $m = V(q)$ for some point in the set over which we are minimized. Therefore, $m > 0$. Supposing for contradiction that $\phi_t(p)$ stays inside $B_\varepsilon(x_0)$ for all $t \geq 0$, we have $\dot{V}(\phi_t(p) \geq m$ for all $t \geq 0$. Hence,

$$V(\phi_t(p)) - V(p) = V(\phi_t(p)) - V(\phi_0(p)) = \int_{s=0}^{t} \dot{V}(\phi_s(p))\, ds \geq mt \to \infty$$

as $t \to \infty$. But since V is continuous, it achieves a maximum on $\overline{B_\varepsilon(x_0)}$—a contradiction. $\square$

Example. Consider the system

$$\begin{aligned} x' &= -2y + yz \\ y' &= x - xz \\ z' &= xy. \end{aligned}$$

and the Jacobian at the origin is

$$J(0) = \begin{pmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The characteristic polynomial is

$$\det \begin{pmatrix} -x & -2 & 0 \\ 1 & -x & 0 \\ 0 & 0 & -x \end{pmatrix} = -x^3 - 2x = -x(x^2 + 2).$$

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$$V = ax^2 + by^2 + cz^2$$

with positive constants a, b, c. We have

$$\dot{V} = 2axx' + 2byy' + 2czz'$$

$$
\begin{aligned}
&= 2ax(-2y + yz) + 2by(x - xz) + 2cz(xy) \\
&= 2(-2a + b)xy + (a - b + c)xyz.
\end{aligned}
$$

Take $a = c = 1$ and $b = 2$, and we get $V = x^2 + 2y^2 + z^2$ with $\dot{V} = 0$. This means that trajectories stay on the ellipsoids that are level sets of V.

Week 10, Monday: Planar systems

Equilibrium points for planar systems

Consider a general planar system

$$\begin{aligned} x' &= P(x, y) \\ y' &= Q(x, y). \end{aligned}$$

By translating, we can assume that any equilibrium point we are interested in sits at the origin.

Here are some types of equilibrium points.

1. The origin is a **center** if there exists $\delta > 0$ such that every trajectory with initial condition in $B_\delta \setminus \{(0,0)\}$ is a closed curve containing $(0,0)$ in its interior.

2. Let $r(t, r_0, \theta_0)$ and $\theta(t, r_0, \theta_0)$ denote the solution to our system in polar coordinates and with initial conditions $r(0) = r_0$ and $\theta(0) = \theta_0$. The origin is a **stable focus** if there exists $\delta > 0$ such that $0 < r_0 < \delta$ and $\theta_0 \in \mathbb{R}$ imply $r(t, r_0, \theta_0) \to (0,0)$ and $|\theta(t, r_0, \theta_0)| \to \infty$ as $t \to \infty$. It is an **unstable focus** if the same holds as $t \to -\infty$.

3. The origin is a **stable node** if there exists $\delta > 0$ such that for $0 < r_0 < \delta$ and $\theta_0 \in \mathbb{R}$, we have $r(t, r_0, \theta_0) \to (0,0)$ as $t \to \infty$ and $\lim_{t\to\infty} \theta(t, r_0, \theta_0)$ exists. In other words, the trajectories approach the origin with a well-defined tangent. It's an **unstable node** if the same holds with $t \to -\infty$. A node is called *proper* if every ray through the origin is tangent to some trajectory.

4. The origin is a **topological saddle** if it is locally homeomorphic to a saddle for a linear system.

5. The origin is a **center-focus** if there exists a sequence of closed solution curves Γ_n with Γ_{n+1} in the interior of Γ_n such that $\Gamma_k \to (0,0)$ as $k \to \infty$ and such that every solution with initial condition between Γ_n and Γ_{n+1} spirals toward either Γ_n or Γ_{n+1} as $t \to \pm\infty$.

Summary of results for hyperbolic equilibria.

Let $x_0 = (0, 0)$ be a hyperbolic equilibrium point and assume that P and Q are continuously differentiable. Then

1. The point x_0 is a topological saddle if and only if the linearized system has a saddle at the origin. (This follows from Hartman-Grobman.)
2. If the linearized system has a center, then x_0 is either a center, a focus, or a center-focus. The case of a center-focus cannot occur if P and Q are *analytic* at x_0, i.e., if they can be expressed as power series that converge in some disc about the origin.
3. If x_0 is a node then the linearized system it's a node or a focus for the nonlinear system. Similarly, if it's a focus for the linearized system, then its a node or focus for the nonlinear system. If f has continous second partials, then if x_0 is a node for the linearized system, it is also a node for the nonlinear system, and similarly for foci. (See our text, Example 5, Section 2.10.)

Example. Here is an example of a center-focus:

$$x' = -y + x\sqrt{x^2+y^2}\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$

$$y' = x + y\sqrt{x^2+y^2}\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right).$$

for $x^2 + y^2 \neq 0$, and with $f(0, 0) = (0, 0)$ where f is the right-hand side of the above. (In particular, it turns out that f is not analytic at the origin.) Changing to polar coordinates gives the system

$$r' = r^2 \sin\left(\frac{1}{r}\right)$$

$$\theta' = 1$$

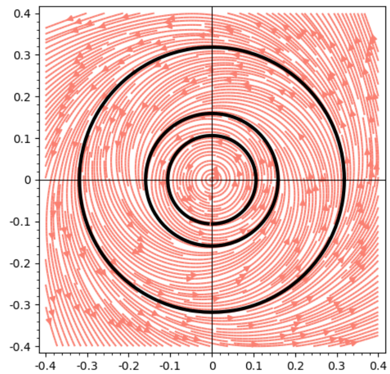
for $r > 0$, and $r' = 0$ for $r = 0$. So $\theta = t + \theta_0$, and if $\sin(1/r) = 0$, i.e., if $r = \frac{1}{n\pi}$ for any $n \in \mathbb{Z}_{>0}$, we have $r' = 0$. So the circles of radius $\frac{1}{n\pi}$ are trajectories. If

$$n\pi < \frac{1}{r} < (n+1)\pi,$$

i.e., if

$$\frac{1}{(n+1)\pi} < r < \frac{1}{n\pi},$$

then $r' < 0$ if n is odd and $r' > 0$ if n is even. Which means the trajectories will either spin inwards or outwards towards one of the circular trajectories. A partial picture appears below:



Nonhyperbolic equilibria. Please set our text, Section 2.11 for a description of possible behaviors for a nonhyperbolic equilibrium point for a two-dimensional system. In particular, please learn the meaning of the following terms: sector, hyperbolic sector, parabolic sector, elliptic sector, saddle-node.

Note the comment on p. 150: if the linearized system is nonzero, the only types of equilibrium points that can occur beside those already mentioned for analytic systems are saddle-nodes, critical points with elliptic domains, and cusps. The book gives examples of each of these:

saddle-node (two hyperbolic sectors, one parabolic sector):

$$\begin{aligned} x' &= x^2 \\ y' &= y \end{aligned}$$

critical point with elliptic domain (one elliptic sector, one hyperbolic sector, two parabolic sectors, four separatrices):

$$\begin{aligned} x' &= y \\ y' &= -x^3 + 4xy \end{aligned}$$

cusp (two hyperbolic sectors, two separatrices):

$$\begin{aligned} x' &= y \\ y' &= x^2 \end{aligned}$$

Week 10, Wednesday: Global theory for nonlinear systems: index theory

GLOBAL THEORY FOR NONLINEAR SYSTEMS: INDEX THEORY

A *Jordan curve* C is the injective continuous image $\gamma\colon S^1 \to \mathbb{R}^2$ of a circle into the plane. Equivalently, it is the continuous image of an interval $\gamma\colon [0,1] \to \mathbb{R}^2$ that is injective on $[0,1)$ and such that $\gamma(0) = \gamma(1)$. The Jordan curve theorem (first conjectured by Bolzano) states that a Jordon curve divides the plane into two connected components. We will impose the further condition that γ be piecewise smooth (continuous derivatives except at a finite number of points).

Let $f(x,y) = (P(x,y), Q(x,y))$ be a smooth vector field in the plane, and let C be a Jordan curve. A *critical point* for f is a point (x_0, y_0) where $f(x_0, y_0) = 0$. (Thus, a critical point would be an equilibrium point for the corresponding system of differential equations.)

Definition. The index $I_f(C)$ of C relative to f is

$$I_f(C) := \frac{\Delta\theta}{2\pi}$$

where $\Delta\theta$ is the change in angle of $f(x,y)$ as (x,y) travels around C counterclockwise.

Exercises.

1. For each of the following vector fields, (i) draw the flow near the origin and draw a circle C containing the origin; (ii) pick some points on C, and draw each point as a vector (with tail at the origin) on a separate picture of $\mathbb{R}^2$; (iii) compute the index:

   (a) $f(x,y) = (-1,-1)$ (b) $f(x,y) = (-x,-y)$

   (c) $f(x,y) = (-y,x)$ (d) $f(x,y) = (-x,y)$.

2. How does the index change in (a)–(d) if f is replaced by $-f$?

3. How would the index change if C were replaced by an ellipse?

Calculation of the index. Let $\gamma(t) = (x(t), y(t))$ be a parametrization of C. By translating, if necessary, we may assume the origin is in the interior of C. Consider the composition of mappings

$$[0,1) \xrightarrow{\gamma} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}^2.$$

We are interested in the change in the angle of $f \circ \gamma$ as t goes from 0 to 1. Write $f \circ \gamma$ in polar coordinates:

$$P(x,y) = r\cos(\theta) \qquad Q(x,y) = r\sin(\theta)$$

where x, y, r, θ are functions of t. Then

$$\begin{aligned} P' &= r'\cos(\theta) - r\theta'\sin(\theta) \\ Q' &= r'\sin(\theta) + r\theta'\cos(\theta), \end{aligned}$$

and it's easy to check that

$$r^2\theta' = PQ' - QP'$$

where $r^2 = P^2 + Q^2$. Therefore, the change in angle is

$$\Delta\theta = \int_{t=0}^{1} \frac{PQ' - QP'}{P^2+Q^2}\,dt = \int_{t=0}^{1} (P,Q)\cdot\left(\frac{Q'}{P^2+Q^2}, -\frac{P'}{P^2+Q^2}\right)\,dt = \oint_C \frac{PdQ - QdP}{P^2+Q^2}.$$

So the index is

$$I_f(C) = \frac{\Delta\theta}{2\pi} = \frac{1}{2\pi}\oint_C \frac{PdQ - QdP}{P^2+Q^2}.$$

To convert to the language of differential forms, let

$$\omega := \frac{x\,dy - y\,dx}{x^2+y^2}$$

be the "flow form" for the circular vector field $\left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ on $\mathbb{R}^2$. Then the index is calculated by integrating the pullback of ω along f over C:

$$I_f(C) = \frac{1}{2\pi}\oint_\gamma f^*\omega.$$

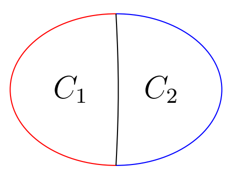
Example. Let $f(x,y) = (-y,x)$, and let C be the unit circle centered at the origin parametrized by $\gamma(t) = (\cos(t), \sin(t))$ for $t \in [0, 2\pi]$. Then

$$f(\gamma(t)) = (-\sin(t), \cos(t)).$$

$$\begin{aligned}
I_f(C) = \frac{\Delta\theta}{2\pi} &= \frac{1}{2\pi}\oint_C \frac{PdQ - QdP}{P^2+Q^2} \\
&= \frac{1}{2\pi}\int_C (P,Q)\cdot\left(\frac{Q'}{P^2+Q^2}, -\frac{P'}{P^2+Q^2}\right) dt \\
&= \frac{1}{2\pi}\int_{t=0}^{1} (-\sin(t),\cos(t))\cdot\left(\frac{-\sin(t)}{(-\sin(t))^2+\cos(t)^2}, -\frac{-\cos(t)}{(-\sin(t))^2+\cos(t)^2}\right) dt \\
&= \frac{1}{2\pi}\int_{t=0}^{2\pi} dt = 1.
\end{aligned}$$

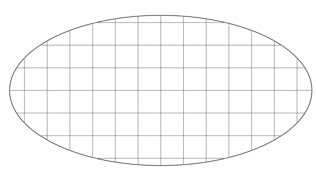
Theorem. If there are no critical points on C or in its interior, then $I_f(C) = 0$.

Proof. **Step 1.** Suppose $C = C_1 + C_2$ as shown below:

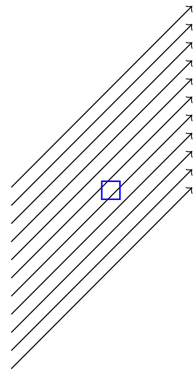


The curves C_1 and C_2 share the vertical middle line. In the calculating of the sum of the indices of f relative to C_1 and to C_2, the contribution from the middle line cancels (imaging traveling along both C_1 and C_2 in the counterclockwise direction). Thus, $I_f(C) = I_f(C_1) + I_f(C_2)$.

Step 2. Next, divide C into a sum of lots of tiny closed curves:



So $C = C_1 + \cdots + C_n$. It suffices to show that $I_f(C_i) = 0$ for all i. Some details and a reference are provided below, but the main idea is that since $f \neq \vec{0}$ on or inside C, by taking C_i small enough, the vector field's angle cannot change much along C_i:



Let X denote C union its interior. Since X is compact, and the component functions P and Q of the vector field are continuous, it follows that P and Q are uniformly continuous. That means that given any $\varepsilon > 0$, we can make the widths of the C_i simultaneously small enough to that P and Q change by a value less then ε on each C_i. Also, since X is compact and f is continuous and nonzero in X, the value of $|f(x, y)|$ attains a nonzero minimum on X. This means that it is possible to take the widths of the C_i simultaneously small enough so that the angle of f on X varies by only a small amount (less than 2π is sufficient). Some details appear in our text, Problem 2, Chapter 3.

The result then follows from Step 1: $I_f(C) = \sum_i I_f(C_i) = \sum_i 0 = 0$. $\square$

Corollary. Let C be a Jordan curve. Suppose there are no critical points on C but that there may be critical points in its interior. Let C' a Jordan curve in the interior of C, and suppose there are no critical points on C', and there are no critical points in the region between C and C'. Then $I_f(C) = I_f(C')$.

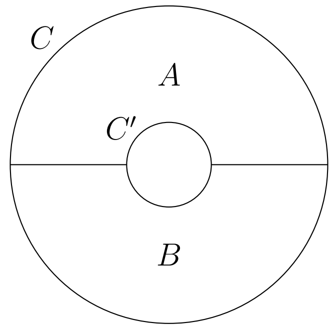
Proof. Referring to the diagram below, let ∂A and ∂B be the Jordan curves forming the boundaries of the closed regions labeled A and B. So both ∂A and ∂B are the boundaries of deformed rectangles. Imagine traveling counterclockwise along these curve, computing their indices. You should see that

$$I_f(\partial A + \partial B) = I_f(C) - I_f(C').$$

However, since there are no critical points in A and none in B, using the previous theorem, we have

$$I_f(\partial A + \partial B) = I_f(\partial A) + I_f(\partial B) = 0.$$

The result follows.



□

Corollary. If C and C' are Jordan curves containing the same finite set of critical points in their interiors, then $I_f(C) = I_f(C')$.

Proof. Let D be a Jordan curve containing all the critical points and contained in the interiors of both C and C'. Then by the previous corollary,

$$I_f(C) = I_f(D) = I_f(C').$$

□

Definition. Let p be an isolated critical point of f. Define the *index of x relative to f* to be

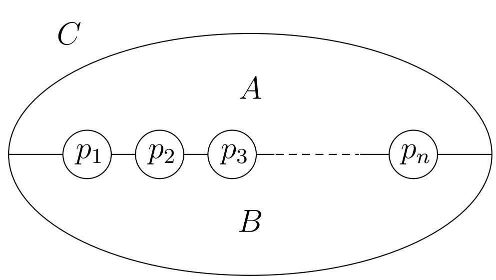
$$I_f(p) := I_f(C)$$

where C is any Jordan curve containing p as its only interior critical point. (This is well-defined from the previous corollary.)

Theorem. Let $p_1, \ldots, p_n$ be the critical points inside C. Then

$$I_f(C) = \sum_{i=1}^{n} I_f(p_i).$$

Proof. The proof is similar to that of our first corollary:



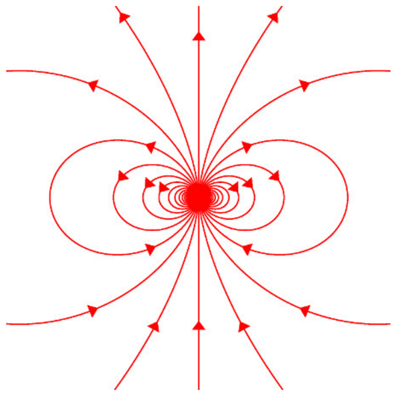
We have

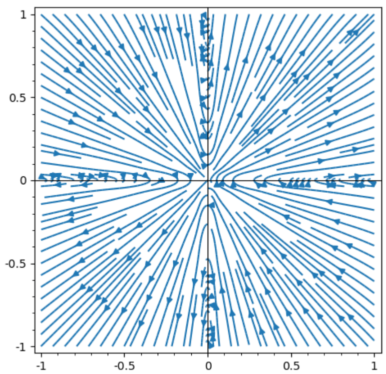
$$0 = I_f(\partial A + \partial B) = I_f(C) - \sum_{i=1}^{n} I_f(p_i).$$

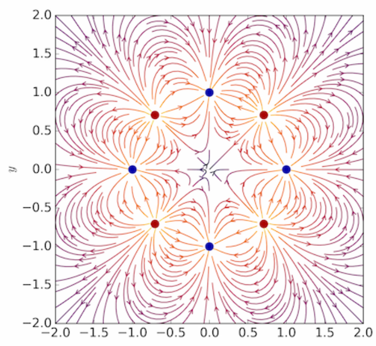
$\square$

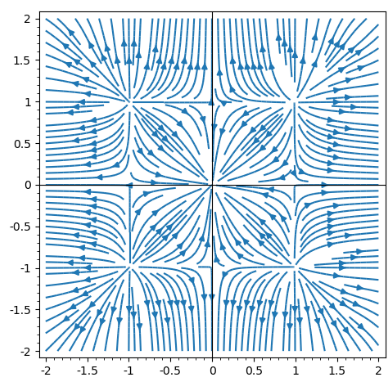
Week 10, Friday: Global theory for nonlinear systems: index theory

Problem 1. Compute the indices for all the critical points pictured below.




1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1


2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2.0 −1.5 −1.0 −0.5 0.0 0.5 1.0 1.5 2.0
x
y


2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

In the remaining problems, all vector fields should be smooth and with isolated singularities.

Problem 2. Draw three vector fields on a sphere: one containing a sink, one containing a center, and one containing a saddle. In addition to the prescribed critical points, your vector field may contain other critical points. Calculate the indices for all critical points. What is the sum of the indices in each of the three instances?

Problem 3. Draw four vector fields on a torus, calculating the indices for all critical points. What is the sum of the indices in each instance?

Problem 4. Draw a vector field on a two-holed donut, calculating the indices for all critical points. What is the sum of the indices?

Problem 5. Draw a vector field on an n-holed torus. Calculate the indices of the critical points, and find the sum of the indices.

Problem 6. Suppose each of the vector fields in Problem 1 was draped over a sphere with the origin at the north pole. Complete the vector field to one on the whole sphere, adding only one more critical point—at the south pole. What would the critical point at the south pole look like? What would its index be? It the type of critical point unique?

Week 11, Monday: Global theory for nonlinear systems: index theory

Our goal now is to formally define the index of a vector field on a surface besides $\mathbb{R}^2$. Let S be a 2-dimensional manifold. So $S = \cap_i U_i$ where the U_i open sets and with homeomorphisms

$$h_i\colon U_i \to V_i \subset \mathbb{R}^2$$

which allow us to think of each U_i as an open subset $V_i \subseteq \mathbb{R}^2$ of the plane. On overlaps $U_i \cap U_j$, the *change of coordinates* mapping $h_j \circ h_i^{-1}$ is differentiable. Recall that the pair (U_i, h_i) is called a *chart* and the collection of charts is an *atlas* . Suppose that $S \subseteq \mathbb{R}^n$. A *vector field* on S is a C^1-mapping $f\colon S \to \mathbb{R}^n$ such that $f(p)$ is tangent to S, so there is a curve $\gamma\colon (-1,1) \to S$ such that $\gamma(0) = p$ and $\gamma'(0) = f(p)$.

To calculate the index of a critical point p of f, i.e., at a point where $f(p) = 0$, we first pick U_i such that $p \in U_i$, and we use h_i to identify f with a vector field on $\mathbb{R}^2$ with critical point $h_i(p)$. In detail, if $q \in U_i$, we pick a curve γ in S passing through q at time 0 and such that $\gamma'(0) = f(q)$. To find the corresponding vector on $\mathbb{R}^2$, we use the composition

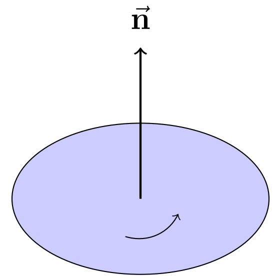
$$(-1,1) \xrightarrow{\gamma} U_i \xrightarrow{h_i} \mathbb{R}^2.$$

The vector in $\mathbb{R}^2$ at the point q will be $(h_i \circ \gamma)'(0)$.

The next thing we need to define the index of a critical point of a vector field on S is an orientation. An *orientation* is a choice of charts so that the Jacobian of each change of coordinates has positive determinant wherever defined. In other words, for all i, j and for all $p \in U_i \cap U_j$, we assume $\det J(h_j \circ h_i^{-1}) > 0$. Then, to define the *index* of a vector field at a critical point p, first choose a chart (U_i, h_i) with $p \in U_i$. Use h_i to translate the vector field f to a vector field $h_{i,*}(f)$ on $V_i = h_i(U_i)$ as described above. Then define the index $I_f(p)$ to be the index of $h_i(p)$ for the vector field $h_{i,*}(f)$ i.e., $I_f(p) := I_{h_{i,*}(f)}(h_i(p))$.

Not all 2-dimensional manifolds are orientable, for instance, a Möbius strip is not orientable, nor is the projective plane $\mathbb{P}^2$ (which contains a Möbius strip).

We now assume that S is compact and oriented. It turns out that in this case, we can take S to be a g-hold donut embedded in $\mathbb{R}^3$. We will take the orientation to be the one determined by the outward-pointing normal vector for S: for a chart at a point p, just take a sufficiently small open piece of the surface containing p, and "flatten is out". We take "counterclockwise" to be the direction for which the right-hand rule gives the outward pointing normal vector:

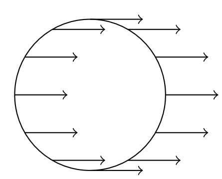


Theorem (Poincaré-Hopf index theorem). Suppose the critical points of the vector field f on S are $p_1, \ldots, p_k$. Then

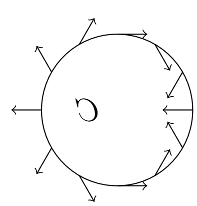
$$\sum_{i=1}^{k} I_f(S) = 2 - 2g.$$

Proof. We first consider the case $g = 0$. So S is an ordinary sphere in $\mathbb{R}^3$. Draw a tiny circle C around a regular point q (i.e., non-critical point). Call the side of C containing q the "inside" of C. We take C tiny enough so that there are no critical points inside C and f is virtually constant around C:

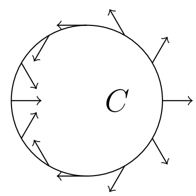
C



Now for the hard part: imagine cutting out the inside of C, stretching C and the remaining part of S so that this remaining part (containing all of the critical points of f) sits in $\mathbb{R}^2$ with C as its boundary. If you are careful with how the vector field morphs under this transformation, you should get the picture:



The vector field now rotates clockwise twice as we go around C. The critical points are now inside the circle in the picture above, so it looks like the sum of their indices should be 2. However, if you are careful, you'll notice that the orientation has reversed (the normal vectors from the sphere are now pointing into the page). So we should flip the picture, to get the usual orientation:



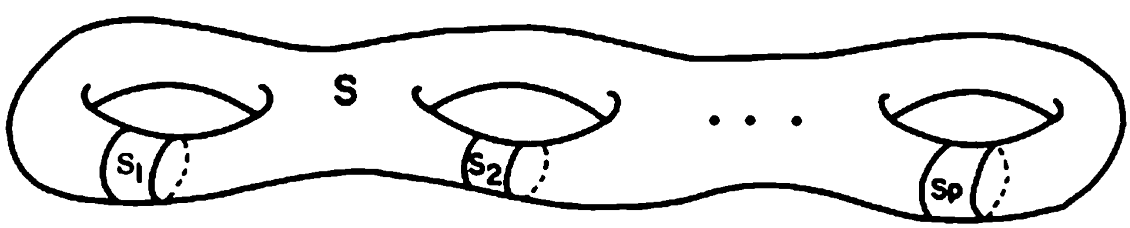
As we travel around the boundary counter-clockwise, the vector field on the boundary rotates twice, again counterclockwise. So the index is still 2 (as opposed to -2).

Thus, the sum of the indices is

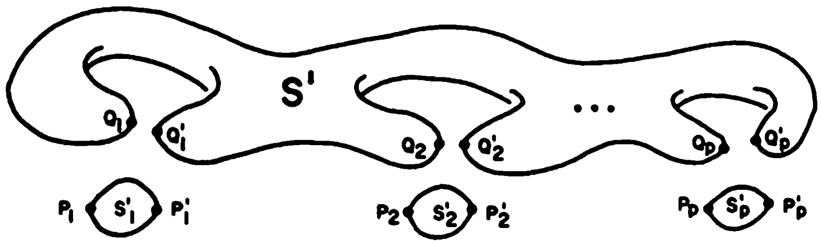
$$2 = 2 - 2 \cdot 0 = 2 - 2g$$

in the case of $S = S^2$.

Now consider a g-holed donut:



The picture is taken from our text, where p is used instead of g. We will keep using g to denote the number of holes. The cylinders $S_1, \ldots, S_g$ are chosen so that they contain no critical points. Next, contract the boundaries of the S_i, morphing the S_i into spheres $S_1', \ldots, S_g'$:



The vector field morphs, too, creating critical points Q_i and Q_i' for $i = 1, \ldots, g$ on the surface S' pictured above, and corresponding critical points P_i and P_i' on the spheres S_i'. Since the vector field at each P_i is the negative of that on Q_i and similarly for P_i' and Q_i', we have

$$I_f(P_i) = I_f(Q_i) \quad \text{and} \quad I_f(P_i') = I_f(Q_i').$$

Since the S_i' are spheres, from our previous work it follows that

$$I_f(P_i) + I_f(P_i') = 2$$

for all i. The surface S' is a sphere. Therefore,

$$I_f(S') = 2.$$

The result follows:

$$\begin{aligned} I_f(S) &= I_f(S') - \sum_{i=1}^{g} (I_f(Q_i) + I_f(Q_i')) \\ &= I_f(S') - \sum_{i=1}^{p} (I_f(P_i) + I_f(P_i')) \\ &= 2 - 2g. \end{aligned}$$

□

Note: The Poincaré-Hopf theorem also holds for non-orientable manifolds. See our text for a proof (that builds on the above proof).

Week 11, Wednesday: Critical points at infinity, and global phase portraits

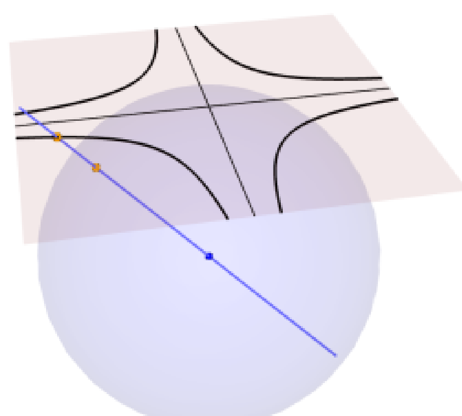
GLOBAL PHASE PORTRAITS

Consider a planar polynomial system:

$$\begin{aligned} x' &= P(x, y) \\ y' &= Q(x, y) \end{aligned} \tag{31.1}$$

where P and Q are polynomials. Our goal not is to look at critical points of this system "at infinity".

Induced flow on the sphere. Imagine our plane as being the $z = 1$ plane in $\mathbb{R}^3$, which we will denote by Π_z, and then project the flow from the plane to the unit sphere S centered at the origin using a line through the center of the sphere:



This will produce a flow on the sphere that naturally extends to its equator. We think of the points on the equator as points at infinity at our plane, and our goal is to examine the critical points there.

To project $(x, y, 1) \in \Pi_z$ to the sphere, we scale it by $Z \in \mathbb{R}$

$$Z(x, y, 1) = (Zx, Zy, Z) =: (X, Y, Z)$$

to get a point on S. The condition is

$$(Zx)^2 + (Zy)^2 + Z^2 = 1.$$

This means

$$Z = \frac{1}{\sqrt{x^2 + y^2 + 1}}.$$

Therefore, the corresponding point on the sphere is

$$(X, Y, Z) = \frac{1}{\sqrt{x^2 + y^2 + 1}}(x, y, 1).$$

Since

$$x = \frac{X}{Z} \quad \text{and} \quad y = \frac{Y}{Z},$$

we may use (31.1) to get

$$\begin{aligned} 0 &= QP - PQ \\ &= Qx' - Py' \\ &= Q\left(\frac{X}{Z}\right)' - P\left(\frac{Y}{Z}\right)' \\ &= Q\left(\frac{X'Z - XZ'}{Z^2}\right) - P\left(\frac{Y'Z - YZ'}{Z^2}\right). \end{aligned}$$

Clearing denominators and regrouping, gives

$$QZX' - PZY' + (PY - QX)Z' = 0$$

To think about this geometrically, we'll write this equation as

$$(QZ, -PZ, PY - QX) \cdot (X', Y', Z') = 0.$$

The solution curve $\gamma(t) = (X(t), Y(t), Z(t))$ has velocity vector

$$\gamma'(t) = (X'(t), Y'(t), Z'(t)),$$

and the above equation says that this curve is perpendicular to the vector

$$N = (QZ, -PZ, PY - QX).$$

In preparation for taking a limit as $Z \to 0$, we consider the functions

$$P(x,y) = P\left(\frac{X}{Z}, \frac{Y}{Z}\right) \quad \text{and} \quad Q(x,y) = Q\left(\frac{X}{Z}, \frac{Y}{Z}\right).$$

As functions of X, Y, and Z, these functions now contain powers of Z as denominators. To clear these denominators, let d be the maximum of the degrees of P and Q, and multiply through by Z^d to get new polynomials:

$$P^* := Z^d P, \quad Q^* := Z^d Q, \quad \text{and} \quad N^* := Z^d N = (Q^* Z, -P^* Z, P^* Y - Q^* X).$$

Since we have only scaled N to get N^*, we still have

$$N^* \cdot \gamma'(t) = (Q^* Z, -P^* Z, P^* Y - Q^* X) \cdot \gamma'(t) = 0.$$

What happens as we approach the equator, i.e., as $Z \to 0$? If $P^* Y - Q^* Z \not\to 0$, then $N^* \to (0, 0, a)$ for some nonzero a. In other words, the vector N^* gets closer and closer to pointing straight up. In turn that means that our trajectory gets closer and closer to running parallel to the equator. So at these points, the induced flow on the equator is just a flow along the equator (not across the equator). This says that the place to look for critical points along the equator are the points $(X, Y, 0)$, where

$$P^* Y - Q^* X = 0. \tag{31.2}$$

Analyzing critical points at ∞. Suppose that using equation (31.2), we find a point $(a, b, 0)$ of interest. Since the point sits on the sphere, at least one of a and b is nonzero. Say $a \neq 0$. We now use central projection to project our flow onto the plane $x = 1$ in $\mathbb{R}^3$, which we denote by Π_x. Taking a point $(x, y, 1) \in \Pi_z$, we scale it to get

$$\left(1, \frac{y}{x}, \frac{1}{x}\right) \in \Pi_x,$$

which we identify with the point

$$\left(\frac{y}{x}, \frac{1}{x}\right) \in \mathbb{R}^2.$$

In other words, we are identifing Π_x with $\mathbb{R}^2$ using these coordinates. Let

$$u := \frac{y}{x} \quad \text{and} \quad v := \frac{1}{x}.$$

From (31.1),

$$x' = \left(\frac{1}{v}\right)' = -\frac{v'}{v^2} = P(x,y) = P\left(\frac{1}{v}, \frac{u}{v}\right)$$

$$y' = \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2} = Q(x,y) = Q\left(\frac{1}{v}, \frac{u}{v}\right).$$

Projecting our point of interest, $(a, b, 0)$ into the plane $x = 1$ gives the point

$$\left(1, \frac{b}{a}, 0\right).$$

So in the u, v-plane representing Π_x, our job is to analyze the point

$$\left(\frac{b}{a}, 0\right)$$

for the system defined by

$$-\frac{v'}{v^2} = P\left(\frac{1}{v}, \frac{u}{v}\right)$$

$$\frac{u'v - uv'}{v^2} = Q\left(\frac{1}{v}, \frac{u}{v}\right).$$

Example. Consider the saddle

$$x' = -x$$
$$y' = y.$$

So $P(x, y) = -x$ and $Q(x, y) = y$. To find the interesting points on the equator, we consider

$$PY - QX = P\left(\frac{X}{Z}, \frac{Y}{Z}\right)Y - Q\left(\frac{X}{Z}, \frac{Y}{Z}\right)X$$

$$= -\frac{X}{Z}Y - \frac{Y}{Z}X = -2\frac{XY}{Z} = 0.$$

Clearing denominators gives

$$2XY = 0.$$

So either $X = 0$ or $Y = 0$. The corresponding points on the equator are

$$(0,1,0) \quad \text{and} \quad (1,0,0).$$

Let's look at $(1,0,0)$, first. We want to project to the $x = 1$ plane. The mapping of interest is

$$(x, y, 1) \rightsquigarrow \left(1, \frac{y}{x}, \frac{1}{x}\right).$$

Let $u = \frac{y}{x}$ and $v = \frac{1}{x}$ and substitute into our system. The first equation in the system says

$$x' = \left(\frac{1}{v}\right)' = -\frac{v'}{v^2} = -x = -\frac{1}{v}.$$

Therefore,

$$v' = v.$$

Continuing with the second equation in the system:

$$\begin{aligned}
y' &= \left(\frac{u}{v}\right)' \\
&= \frac{u'v - uv'}{v^2} \\
&= \frac{u'v - uv}{v^2} && \text{(since } v' = v\text{)} \\
&= \frac{u' - u}{v} \\
&= y = \frac{u}{v}.
\end{aligned}$$

Therefore, $u' - u = u$, and so

$$u' = 2u.$$

Thus, at the point $(1,0,0)$ on the equator, our system looks like the system

$$\begin{aligned}
u' &= 2u \\
v' &= v,
\end{aligned}$$

which is a **source**.

Now let's look at the other interesting point on the equator, $(0,1,0)$. The relevant mapping is

$$(x, y, 1) \rightsquigarrow \left(\frac{x}{y}, 1, \frac{1}{y}\right).$$

Now let $u = \frac{x}{y}$ and $v = \frac{1}{y}$ and consider the point $(1,0)$. Plug these into the system:

$$y' = \left(\frac{1}{v}\right)' = -\frac{v'}{v^2} = y = \frac{1}{v}.$$

Thus,

$$v' = -v.$$

Next,

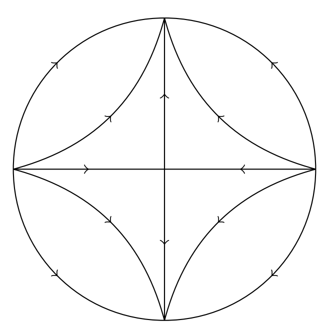
$$\begin{aligned} x' &= \left(\frac{u}{v}\right)' \\ &= \frac{u'v - uv'}{v^2} \\ &= \frac{u'v + uv}{v^2} && \text{(since } v' = -v\text{)} \\ &= \frac{u' + u}{v} \\ &= -x = -\frac{u}{v}. \end{aligned}$$

It follows that $u' = -2u$. So the system becomes

$$\begin{aligned} u' &= -2u \\ v' &= -v, \end{aligned}$$

a **sink**.

Global phase portrait. To get the global phase portrait for a planar system, project the flow onto the upper-hemisphere of the unit sphere, using the process described above, then position yourself way above the north pole, and look down. For the saddle we just considered this looks like:



(If we identify antipodal points on the boundary, we'd get a flow on $\mathbb{P}^2$, the projective plane.)

Week 11, Friday: Critical points at infinity, and global phase portraits

Recall the notation from last time. We are studying a planar system

$$x' = P(x, y) \tag{32.1}$$
$$y' = Q(x, y) \tag{32.2}$$

where P and Q are polynomials. We embedded our system in the plane $z = 1$ in $\mathbb{R}^3$ and projected the flow along lines centered at the origin onto the unit sphere S centered at the origin. This flow induced a flow along the equator of the sphere. We are interested in critical points of this flow along the equator, where $z = 0$. To find those we saw that we should ...

Step 1. Clear denominators in the equation

$$yP\left(\frac{x}{z}, \frac{y}{z}\right) - xQ\left(\frac{x}{z}, \frac{y}{z}\right) = 0,$$

and then set $z = 0$.

The result is several points of the form $(a, b, 0)$ on the equator of the sphere.

Step 2. To analyze these, we will project the flow on the sphere to either the plane $x = 1$ or the plane $y = 1$. If a, b are both nonzero, then either plane will do. If $a = 0$, then the point in question is $(0, 1, 0)$, and we'd need to project to the plane $y = 1$, and if $b = 0$, the point is $(1, 0, 0)$, and we'd need to project to $x = 1$. We'll consider these cases separately:

Projection to the plane $x = 1$. We project the point $(x, y, 1)$ along a line through the origin, i.e., we scale this point, to get a point on the plane $x = 1$:

$$(x, y, 1) \rightsquigarrow \left(1, \frac{y}{x}, \frac{1}{x}\right).$$

Let

$$u := \frac{y}{x} \quad \text{and} \quad v := \frac{1}{x}.$$

We use the coordinates to identify the plane $x = 1$ with the ordinary plane $\mathbb{R}^2$. It follows that

$$x = \frac{1}{v} \quad \text{and} \quad y = \frac{u}{v}.$$

Plug these into system 32.1:

$$\begin{aligned} x' = P(x,y) \quad &\Rightarrow \quad \left(\frac{1}{v}\right)' = P\left(\frac{1}{v}, \frac{u}{v}\right) \\ &\Rightarrow \quad v' = -v^2 P\left(\frac{1}{v}, \frac{u}{v}\right) \end{aligned}$$

Similarly,

$$\begin{aligned} y' = Q(x,y) \quad &\Rightarrow \quad \left(\frac{u}{v}\right)' = Q\left(\frac{1}{v}, \frac{u}{v}\right) \\ &\Rightarrow \quad u'v - uv' = v^2 Q\left(\frac{1}{v}, \frac{u}{v}\right) \\ &\Rightarrow \quad u'v = v^2 Q\left(\frac{1}{v}, \frac{u}{v}\right) + uv' \\ &\Rightarrow \quad u'v = v^2 Q\left(\frac{1}{v}, \frac{u}{v}\right) - uv^2 P\left(\frac{1}{v}, \frac{u}{v}\right) \\ &\Rightarrow \quad u' = v\left(Q\left(\frac{1}{v}, \frac{u}{v}\right) - uP\left(\frac{1}{v}, \frac{u}{v}\right)\right). \end{aligned}$$

So the system in the u, v-plane is

$$\begin{aligned} u' &= v\left(Q\left(\frac{1}{v}, \frac{u}{v}\right) - uP\left(\frac{1}{v}, \frac{u}{v}\right)\right) \qquad (32.3) \\ v' &= -v^2 P\left(\frac{1}{v}, \frac{u}{v}\right). \end{aligned}$$

The problem is that it is likely this system is not defined where $v = 0$ (at the equator). To get the induced flow on the equator, we need to clear denominators (thus, scaling the vector field but not changing its direction at any point). Define

$$d := \max\{\deg P, \deg Q\}$$

To clear denominators we scale the vector field in (32.3) by v^{d-1} to get the system

$$\boxed{\begin{aligned} u' &= v^d\left(Q\left(\frac{1}{v},\frac{u}{v}\right) - uP\left(\frac{1}{v},\frac{u}{v}\right)\right) \\ v' &= -v^{d+1}P\left(\frac{1}{v},\frac{u}{v}\right). \end{aligned}}$$

We analyze the point $\left(\frac{b}{a},0\right)$, since this is the point corresponding to $(a,b,0)$ in the u,v-plane.

Important points: The right side of the system (32.3) defines the vector field whose trajectories we would like to determine. What effect does scaling that vector field by v^{d-1} have on the solution trajectories? The vector field gives the tangent vector for a solution trajectory. So one effect is to scale the speed of the trajectory by the magnitude $|v|^{d-1}$. Note that $v = 1/x$ where x comes from the point $(x,y,1)$ in the $z = 1$ plane. As we go "out to infinity" in the $z = 1$ plane, by taking x larger, the scaling factor $|v^{d-1}|$ decreases in magnitude. We have chosen d just write so that the resulting vector field does not blow up on the equator and is also not identically the zero on the equator.

What about the direction of the trajectory? Since both components of the vector field are scaled the same amount, there are two choices: (i) if $v^{d-1} > 0$, the direction is the same, and (ii) if $v^{d-1} < 0$, the direction is reversed. Next, what significance does this have for analyzing critical points at the equator? Suppose we are interesting in a trajectory corresponding containing a point $(x,y,1)$ in the original $z = 1$ plane. If $x > 0$, then since $v = 1/x > 0$, it follows that $v^{d-1} > 0$, and the direction does not change. On the other hand, if $x < 0$, then $v = 1/x < 0$. If d is odd, then $v^{d-1} > 0$, and if d is even, then $v^{d-1} < 0$. So in the latter case, in which d is even, the direction of the vector field and hence the direction of it solution trajectories is reversed.

Projection to the plane $y = 1$. By a similar analysis (which will be assigned for homework), if $b \neq 0$, we can project to the $y = 1$ plane and derive an analogous system of equations:

$$\boxed{\begin{aligned} u' &= v^d\left(P\left(\frac{u}{v},\frac{1}{v}\right) - uQ\left(\frac{u}{v},\frac{1}{v}\right)\right) \\ v' &= -v^{d+1}Q\left(\frac{u}{v},\frac{1}{v}\right). \end{aligned}}$$

We are interested in the point $\left(\frac{a}{b},0\right)$ in this plane.

Global phase portrait A couple of lectures ago, we introduced the global phase portrait of a planar system. It is the central projection of the flow of the vector field onto the top half of the sphere. In order to compute it, find and analyze all critical points of the planar system. Next, find all critical points of the system at infinity. These come in antipodal pairs: $(a, b, 0)$ and $(-a, -b, 0)$. Without loss of generality, suppose $a > 0$. Then we analyze the scaled system, scaling by v^{d-1} with $v = 1/x$ in the plane $x = 1$ at the point $\left(\frac{b}{a}, 0\right)$. At the antipodal point, we analyze the same system but scaled by $(-1/x)^{d-1}$ and at the point $\left(\frac{-b}{-a}, 0\right)$, i.e., at the same point. This means the systems at antipodal points are either the same when projected to the $x = 1$ plane up to a possible reversal of directions (which happens exactly in the case d is even.

Exercises

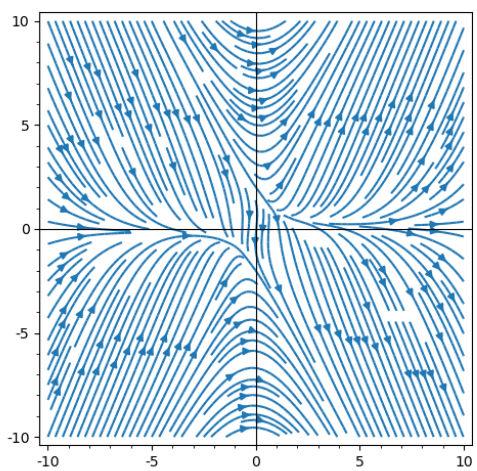
In the following exercises, we will analyze the critical points, both finite and at ∞ for the system

$$\begin{aligned} x' &= x^2 + y^2 - 1 \\ y' &= 5xy - 5. \end{aligned} \tag{32.4}$$

Problem 1. Find all critical points of the system, including critical points at ∞.

Problem 2. Analyze each point at ∞ by projecting to the plane $x = 1$. Draw the flow in the plane $x = 1$.

Problem 3. Try to reconcile your results from Problem 2 with the flow of the original system displayed below:



Try to draw a global phase portrait.

Week 12, Monday: Critical points at infinity, and global phase portraits

Today we will do more examples of global phase portraits. First we recall the method.

We are studying a planar system

$$x' = P(x, y) \tag{33.1}$$
$$y' = Q(x, y) \tag{33.2}$$

where P and Q are polynomials. Embedded the system in the plane $z = 1$ in $\mathbb{R}^3$ and project the flow along lines centered at the origin onto the unit sphere S centered at the origin. This flow induced a flow along the equator of the sphere. We are interested in critical points of this flow along the equator, where $z = 0$. The correspond to *critical points at infinity* in our original planar system.

Step 1. Let $d := \max\{\deg P, \deg Q\}$. Clear denominators in the equation

$$yP\left(\frac{x}{z}, \frac{y}{z}\right) - xQ\left(\frac{x}{z}, \frac{y}{z}\right) = 0,$$

and then set $z = 0$. Note: we clear denominators by multiplying the above expression through by z^d.

The result is several pairs of antipodal points of the form $(a, b, 0), (-a, -b, 0)$ on the equator of the sphere. The critical points occur among these.

Step 2. Projection to the plane $x = 1$: If $a \neq 0$, to analyze the pair of antipodal points $(a, b, 0)$ and $(-a, -b, 0)$, we may assume $a > 0$. To determine the behavior of our spherical system at $(a, b, 0)$, analyze the point $\left(\frac{b}{a}, 0\right)$ for the system

$$\boxed{\begin{aligned} u' &= v^d\left(Q\left(\frac{1}{v}, \frac{u}{v}\right) - uP\left(\frac{1}{v}, \frac{u}{v}\right)\right) \\ v' &= -v^{d+1}P\left(\frac{1}{v}, \frac{u}{v}\right). \end{aligned}}$$

The behavior at the antipodal point $(-a, -b, 0)$ will be the same except if d is even, the direction of flow is reversed. (See the previous lecture for an explanation.)

Projection to the plane $y = 1$: If $a \neq 0$, then if $(a, b, 0) = (0, b, 0)$ is to be on the sphere, we must have $b = \pm 1$. The behavior at $(0, 1, 0)$ is determined by the behavior of the point $(0, 0)$ in the system

$$\boxed{\begin{aligned} u' &= v^d \left(P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right)\right) \\ v' &= -v^{d+1} Q\left(\frac{u}{v}, \frac{1}{v}\right). \end{aligned}}$$

The behavior at $(0, -1, 0)$ is the same except d is even, the direction of the flow is reversed.

Global phase portrait. Imagine looking down from a point way up on the z-axis at the flow we projected to the sphere. Below us, we see a flow on a disc whose boundary is the equator of the sphere. That's the global phase portrait: the projection of the flow on the upper hemisphere of the sphere down to the $z = 0$ plane. By analyzing the behavior of our planar system at all of its equilibrium points, including those at infinity, we show get a good *qualitative* understanding of the flow of the system from its global phase portrait.

Exercises from last time. Consider the system

$$\begin{aligned} x' &= x^2 + y^2 - 1 \\ y' &= 5xy - 5. \end{aligned}$$

Problem 1. Find all critical points of the system, including critical points at ∞.

Solution. Find the critical points at infinity:

$$\begin{aligned} yP\left(\frac{x}{z}, \frac{y}{z}\right) - xQ\left(\frac{x}{z}, \frac{y}{z}\right) &= y\left(\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 - 1\right) - x\left(5\left(\frac{x}{z}\right)\left(\frac{y}{z}\right) - 5\right) \\ &= \frac{x^2y + y^3 - yz^2 - 5x^2y + 5xz^2}{z^2} \\ &= \frac{-4x^2y + y^3 - yz^2 + 5xz^2}{z^2} \end{aligned}$$

$$= 0.$$

Clear denominators:
$$-4x^2y + y^3 - yz^2 + 5xz^2 = 0.$$

Set $z = 0$ to get
$$-4x^2y + y^3 = y(-4x^2 + y^2) = 0.$$

So the points $(x, y, 0)$ at infinity (along the equator of the sphere) occur where $y = 0$ or where $y = \pm 2x$. So we need to consider the three points

$$(1, 0, 0), \quad \frac{1}{\sqrt{5}}(1, \pm 2, 0),$$

and their antipodes. In this case, since $d = \max\{\deg P, \deg Q\} = 2$ is even, the behavior of a pair of antipodes is the same except that the flow is reversed.

Problem 2. Analyze each point at ∞ by projecting to the plane $x = 1$. Draw the flow in the plane $x = 1$.

Solution. We consider the system

$$u' = v^2\left(\frac{5u}{v^2} - 5 - u\left(\frac{1}{v^2} + \frac{u^2}{v^2} - 1\right)\right) = 4u - 5v^2 - u^3 + uv^2$$

$$v' = -v^3\left(\frac{1}{v^2} + \frac{u^2}{v^2} - 1\right) = -v - u^2v + v^3.$$

We are interested in the critical points $(0, 0)$ and $(\pm 2, 0)$ for the system

$$\begin{aligned} u' &= 4u - 5v^2 - u^3 + uv^2 \\ v' &= -v - u^2v + v^3. \end{aligned}$$

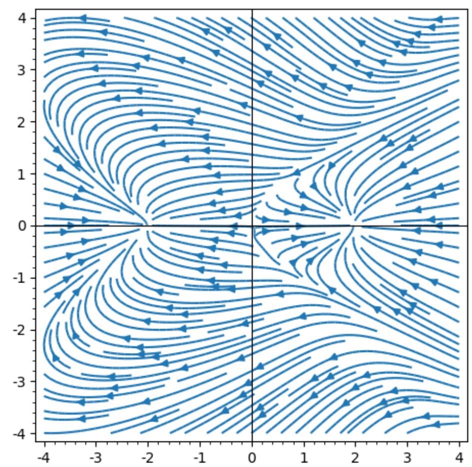
The Jacobian matrix for the right-hand side is

$$Jf := \begin{pmatrix} 4 - 3u^2 + v^2 & -10v + 2uv \\ -2uv & -1 - u^2 + 3v^2 \end{pmatrix}.$$

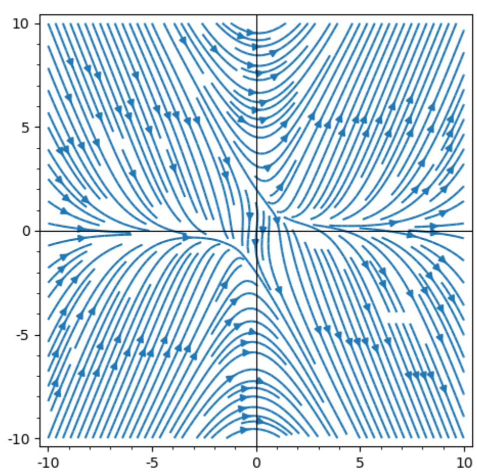
We have
$$Jf(0, 0) = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad J(\pm 2, 0) = \begin{pmatrix} -8 & 0 \\ 0 & -5 \end{pmatrix}.$$

Thus, we get a saddle at $(0, 0)$ and a sinks at $(\pm 2, 1)$.

Here is the flow given by this system in the $x = 1$ plane:



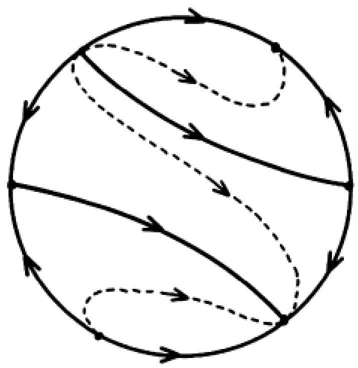
Problem 3. Try to reconcile your results from Problem 2 with the flow of the original system displayed below:

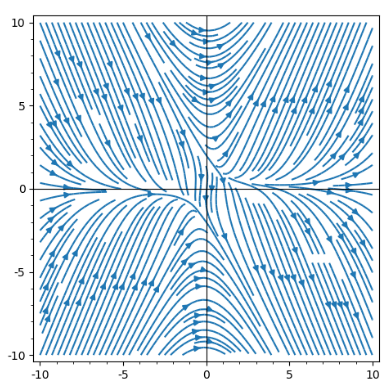


Try to draw a global phase portrait.

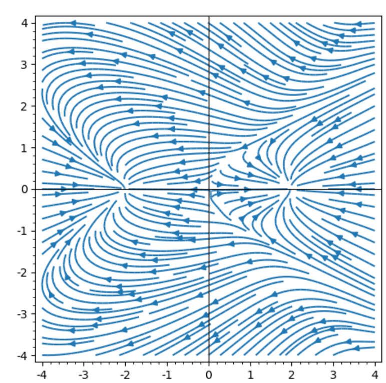
Solution. Notice in the original system, the almost parallel trajectories heading off to the northeast as a slope of about 2. If we consider draping the plane over an underlying sphere, these trajectories will converge to the critical point $(1, 2, 0)$ on the equator. Similarly, the trajectories heading to the southeast with a slope of approximately -2 will converge to $(1, -2, 0)$. The trajectories streaming in on the left from the northwest and southwest are coming from the antipodal points $(-1, 2, 0)$ and $(-1, -2, 0)$.

Global phase portrait:





original system



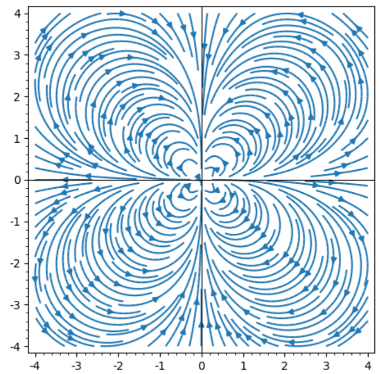
$x = 1$ plane projection

We finish with a couple more examples of global phase portraits.

Example. Consider the system

$$x' = x^3 - 3xy^2$$
$$y' = 3x^2y - y^3.$$

The flow is



To find the interesting points at infinity (along the equator of the sphere) we compute

$$yP\left(\frac{x}{z}, \frac{y}{z}\right) - xQ\left(\frac{x}{z}, \frac{y}{z}\right) = 0,$$

then clear denominators and set $z = 0$ to get

$$0 = -2x^3y - 2xy^3 = -2xy(x^2 + y^2).$$

There are no points on the equator for which $x^2 + y^2 = 0$, i.e., for which $x = y = 0$. So the solutions occur when on of x and y is 0 and the other is nonzero. That gives two pair of antipodal points to check: $\pm(1, 0, 0)$ and $\pm(0, 1, 0)$. To check $(1, 0, 0)$ we project to the $x = 1$ plane. The system is

$$\begin{aligned} u' &= 2u + 2u^3 \\ v' &= -v + 3u^2v \end{aligned}$$

Linearized at the point $(0, 0)$, the system becomes

$$\begin{aligned} u' &= 2u \\ v' &= -v, \end{aligned}$$

which is a saddle. Since $d = 3$ is odd, the behavior at the antipodal point $(-1, 0, 0)$ is the same.

For the point $(0, 1, 0)$ we project to the plane $y = 1$. The system is

$$\begin{aligned} u' &= -2u - 2u^3 \\ v' &= v - 3u^2v \end{aligned}$$

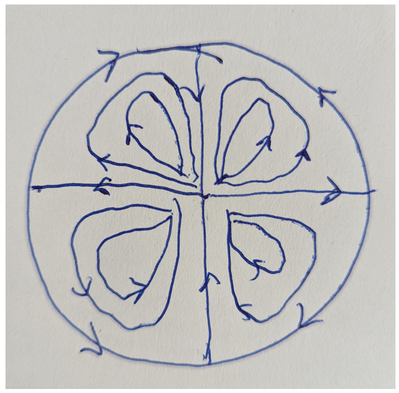
Linearized at the point $(0, 0)$, the system becomes

$$u' = -2u$$

$$v' = v,$$

another saddle. The antipodal point $(0, -1, 0)$ is similar.

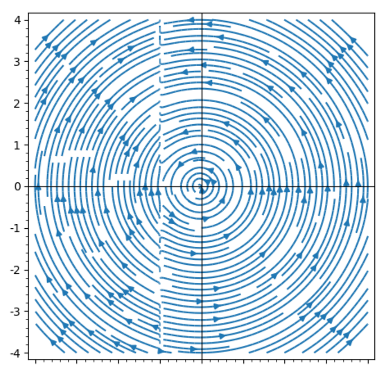
The global phase portrait is:



Example. Consider the system

$$x' = -y - xy$$
$$y' = x + x^2.$$

The flow is



To find the interesting points at infinity (along the equator of the sphere) we compute

$$yP\left(\frac{x}{z}, \frac{y}{z}\right) - xQ\left(\frac{x}{z}, \frac{y}{z}\right) = 0,$$

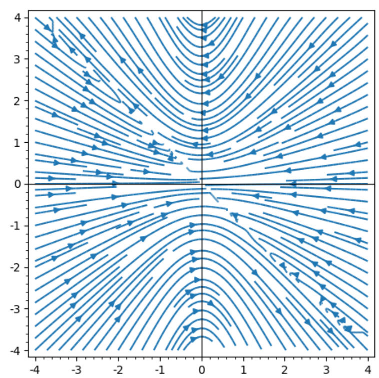
then clear denominators and set $z = 0$ to get

$$0 = -x^3 - xy^2 = -x(x^2 + y^2)$$

So we need $x = 0$. That means we are interested in the points $\pm(0, 1, 0)$ on the equator. Projecting to the plane $y = 1$, we get the system

$$\begin{aligned} u' &= -u - v - u^3 - u^2 v \\ v' &= -u^2 v - uv^2, \end{aligned}$$

which looks like



Note the line of critical points in the planar flow diagram and how those transform when that flow is projected onto the sphere and then onto the plane $x = 1$.

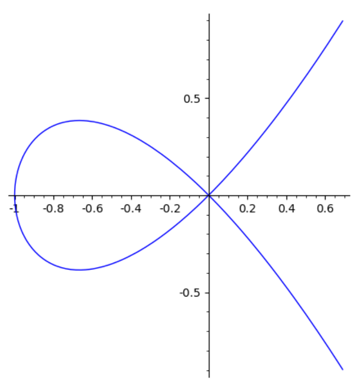
Week 12, Wednesday: Resolution of singularities

BLOWING UP CRITICAL POINTS

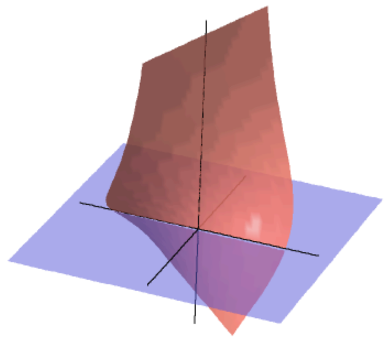
We would like to describe the process of *blowing up* a singularity in an algebraic curve. Consider the nodal cubic curve C

$$y^2 = x^3 + x^2 = x^2(x+1)$$

in the plane. By this we mean the set $\{(x,y) \in \mathbb{R}^2 : y^2 = x^3 + x^2\}$:



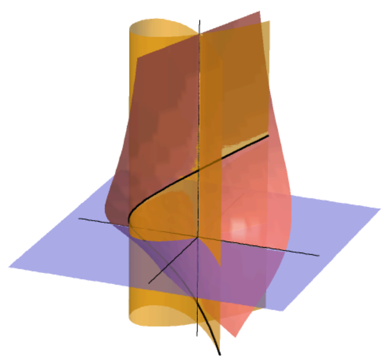
To blow up the singularity at $(0,0)$, we embed the curve in the $z = 0$ plane in $\mathbb{R}^3$. Next consider the "corkscrew" surface B in $\mathbb{R}^3$ defined by the equation $y = zx$. Intersecting this surface with the plane $z = m$ for each constant m, we get the line $y = mx$ with slope m:



Define the cylinder over the curve C:

$$\mathrm{Cyl}(C) := \left\{(x, y, z) \in \mathbb{R}^3 : (x, y) \in C\right\},$$

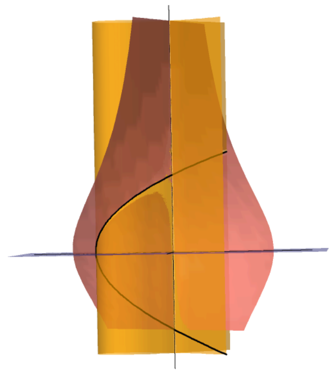
and imagine the intersection $\mathrm{Cyl}(C) \cap B$ of this cylinder with the corkscrew surface:



Algebraically, the intersection is the set of solutions (x, y, z) to the equations

$$y^2 = x^3 + x^2 \quad \text{and} \quad y = zx.$$

Substituting $y = zx$ into the first equation. We see that either the solution is the z-axis, and the curve on the surface $y = zx$ satisfying $z^2 = x + 1$. The z-axis is called the *exceptional divisor* and corresponds to the singularity of C. Project the curve on the corkscrew surface to the $y = 0$ plane to get the curve $z^2 = x + 1$, a parabola:

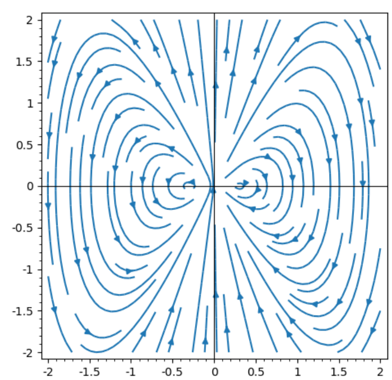


This parabola is the blow up of the nodal cubic at the origin. In intersection of the cylinder over the nodal cubic C with the corkscrew surface, each point of C besides the origin get raised to a height according to the slope of the line between it and the origin. The singularity at the origin gets separated into two points on the corkscrew surface according to its two tangent directions, resolving (desingularizing) the singularity.

We would like to extend this technique to analyze critical points of planar polynomial systems of differential equations. For example, consider the system

$$\begin{aligned} x' &= xy \\ y' &= y^2 - x^4. \end{aligned}$$

The phase portrait is



By inspection, the linearization at the critical point $(0,0)$ is the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

which clearly doesn't tell us much about the shape of the flow around the critical point.

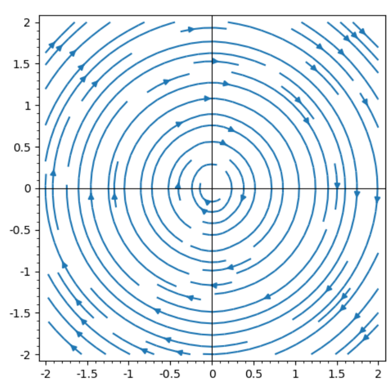
We blow up the critical point using the equation $y = zx$ for the corkscrew surface. We get $y' = z'x + zx'$, which implies

$$\begin{aligned} z' &= \frac{y' - zx'}{x} \\ &= \frac{(y^2 - x^4) - z\,(xy)}{x} \\ &= \frac{(z^2x^2 - x^4) - z^2x^2}{x} \\ &= -x^3. \end{aligned}$$

Notice that by canceling x in the calculation above, we have an equation that is defined at $x = 0$. We've filled in a hole that way. (Recall that in the blow up process, we threw away the z-axis in the intersection of the cylinder with the corkscrew surface). Next,

$$x' = xy = x\,(zx) = x^2z$$

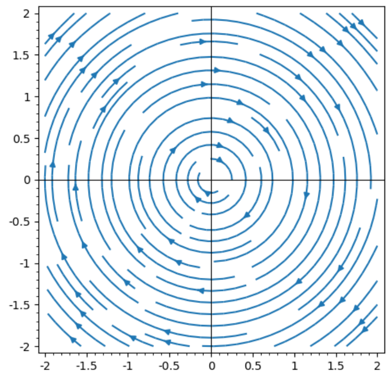
So our system at this point is defined by the vector field $f(x, z) = (x^2z, -x^3)$:



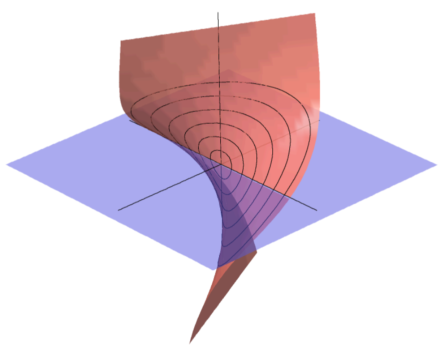
Again, the linearization of this vector field at the origin is given by the zero matrix. However, scaling the vector field by $1/x^2$ at points where $x \neq 0$ gives the vector field $(z, -x)$, which is also defined at $x = 0$: Our final system is

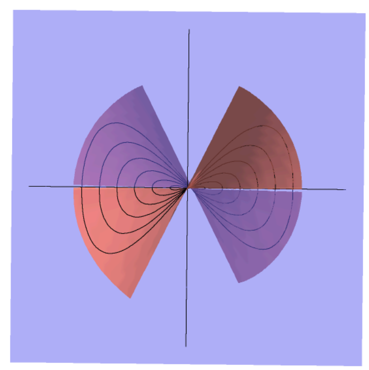
$$\begin{aligned} x' &= z \\ z' &= -x, \end{aligned}$$

which is a center:



To get back out original vector field, imagine this center wrapped along the corkscrew surface, before projecting to the x, z-plane. Take the flow we would get on the corkscrew surface and project that back down to the x, y-plane:





Week 12, Friday: Hamiltonian systems

HAMILTONIAN SYSTEMS

Let $E \subseteq \mathbb{R}^{2n}$ be an open subset, and let $H\colon E \to \mathbb{R}$ be a function in $C^2(E)$, i.e., a function whose second partials exist and are continuous. We will write $H = H(x, y)$ where $x, y \in \mathbb{R}^n$. The system

$$x' = (x_1', \dots, x_n') = H_y := \frac{\partial H}{\partial y} = \left(\frac{\partial H}{\partial y_1}, \dots, \frac{\partial H}{\partial y_n}\right)$$

$$y' = (y_1', \dots, y_n') = -H_x := -\frac{\partial H}{\partial x} = -\left(\frac{\partial H}{\partial x_1}, \dots, \frac{\partial H}{\partial x_n}\right),$$

is called a *Hamiltonian system* with n degrees of freedom. The function H is called the *Hamiltonian* or *total energy* of the system.

Theorem 1. (Conservation of energy.) For a Hamiltonian system, the total energy H is constant along trajectories.

Proof. Consider a solution trajectory $\gamma(t) = (x(t), y(t))$ in $\mathbb{R}^{2n}$. By the chain rule,

$$\begin{aligned}
\frac{d}{dt}H(\gamma(t)) &= \nabla H \cdot \gamma' \\
&= \frac{\partial H}{\partial x} \cdot x' + \frac{\partial H}{\partial y} \cdot y' \\
&= \frac{\partial H}{\partial x} \cdot \frac{\partial H}{\partial y} - \frac{\partial H}{\partial y} \cdot \frac{\partial H}{\partial x} \\
&= 0.
\end{aligned}$$

□

This result means that the solutions lie on level sets for H.

Example. Let $H(x,y) := y\sin(x)$, and consider the Hamiltonian system with one degree of freedom

$$\begin{aligned} x' &= H_y = \sin(x) \\ y' &= -H_x = -y\cos(x), \end{aligned}$$

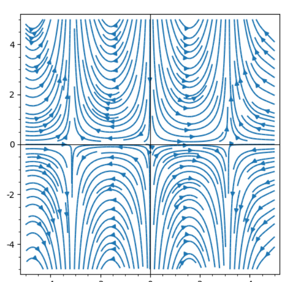
where the x and y subscripts on H denote partial derivatives. We find the critical points:

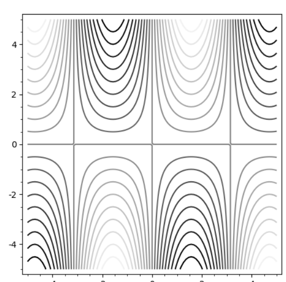
$$x' = y' = 0 \quad \Rightarrow \quad \sin(x) = y\cos(x) = 0 \quad \Rightarrow \quad x = n\pi \text{ and } y = 0,$$

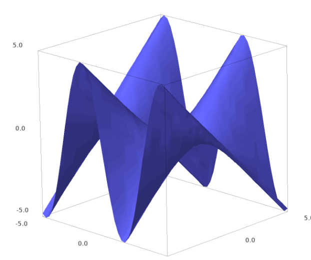
for $n \in \mathbb{Z}$. Letting $f(x,y) = (\sin(x), -y\cos(x))$, the linearizations at these critical points are

$$\begin{aligned} \begin{pmatrix} x' \\ y' \end{pmatrix} = Df(n\pi,0)\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \cos(x) & 0 \\ y\sin(x) & -\cos(x) \end{pmatrix}\Bigg|_{(n\pi,0)} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= (-1)^n \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned}$$

Therefore, the critical points are all topological saddles. Here are pictures of the flow of the system, a contour plot of H (which shows the level sets), and a graph of H:







Critical points. The critical points of a Hamiltonian system $x' = H_x$, $y' = H_y$ occur where where all of the partials of H vanish, i.e., at the critical points for the function H. These are the points where the graph of H,

$$\text{graph}(H) := \left\{ (x,y,H(x,y)) \subset \mathbb{R}^{2n+1} : (x,y) \in E \right\},$$

has a “horizontal tangent space”, i.e., where the tangent space is given by setting the last coordinate equal to zero. (To parametrize the tangent space, imagine the Jacobian of $(x,y) \to (x,y,H(x,y))$. It is the $2n \times 2n$ identity matrix with an appended row consisting of the partials of H. The columns of this matrix span the tangent space.)

At a critical point p, the geometry of H is determined by its second partials (if these don't also vanish). For the purpose of determining this geometry, by translation, we may assume $p = 0$, the origin, and $H(0) = 0$. Then the second-order Taylor polynomial for H will be

$$Q(x, y) = \frac{1}{2}\frac{\partial^2 H}{\partial x_1^2}(0)x_1^2 + \frac{\partial^2 H}{\partial x_1 \partial x_2}(0)x_1 x_2 + \cdots + \frac{1}{2}\frac{\partial^2 H}{\partial y_n^2}(0)y_n^2.$$

By completing squares and making a linear change of coordinates (or appealing to the spectral theorem for real symmetric matrices), we can transform Q into a function of the form:

$$\widetilde{Q} = v_1^2 + \cdots + v_k^2 - v_{k+1}^2 - \cdots - v_r^2$$

where the new coordinates are $v_1, \ldots, v_r, \ldots, v_{2n}$. The number of pluses and minuses turns out to not depend on the choice of change of coordinates and is the crucial geometric information.

Example. In our earlier example, $H(x, y) = y\sin(x)$, the critical points were found to be $(n\pi, 0)$ for $n \in \mathbb{Z}$. To compute the second-order Taylor polynomial at each of these points, we first compute

$$H_{xx} = -y\sin(x), \quad H_{xy} = \cos(x), \quad H_{yy} = 0.$$

So the second-order approximation of H is

$$\begin{aligned} T(x, y) &= H(n\pi, 0) + \frac{1}{2}H_{xx}(n\pi, 0)(x - n\pi)^2 + H_{xy}(n\pi, 0)(x - n\pi)y + \frac{1}{2}H_{yy}(n\pi, 0)y^2 \\ &= (-1)^n(x - n\pi)y. \end{aligned}$$

Letting

$$u := \frac{1}{2}\left(y + (x - n\pi)\right) \quad \text{and} \quad v := \frac{1}{2}(y - (x - n\pi)),$$

we get

$$u - v = x - n\pi \quad \text{and} \quad u + v = y.$$

Using this change of coordinates, the Taylor polynomial becomes

$$\tilde{T} := (-1)^n(u - v)(u + v) = \pm(u^2 - v^2),$$

and the graph of $\tilde{T}$ is a saddle.

Corollary. Let $p \in \mathbb{R}^{2n}$. Suppose there is a solution $\gamma(t) = (x(t), y(t))$ such that $\gamma(0) \neq p$ but such that $\gamma(t) \to p \in \mathbb{R}^{2n}$ as either $t \to \infty$ or $t \to -\infty$. Then p is not a strict minimum or maximum of H.

Proof. Suppose $\lim_{t\to\infty} \gamma(t) = p$. Using Theorem 1, we know that $H(\gamma(t))$ is constant. Therefore, for all t, we have $H(\gamma(0)) = H(\gamma(t))$. Taking the limit at $t \to \infty$ and using the fact that H is continuous, we get

$$H(\gamma(0)) = \lim_{t\to\infty} H(\gamma(t)) = H(\lim_{t\to\infty} \gamma(t)) = H(p).$$

Thus, in any neighborhood of p, there is a path along which H is constant with value $H(p)$. A similar argument holds in the case $\gamma(t) \to p$ as $t \to -\infty$. $\square$

Theorem 2. Consider a Hamiltonian system with one degree of freedom and total energy function $H(x, y)$. Suppose that H is analytic (i.e., it can be written as a convergent power series at every point in its domain). Then every nondegenerate critical point of the system (points where the linearization has two nonzero eigenvalues) is either a topological saddle or a center. It's a topological saddle if and only if its a saddle for H and it's a center if and only if it's a strict local minimum or maximum for H.

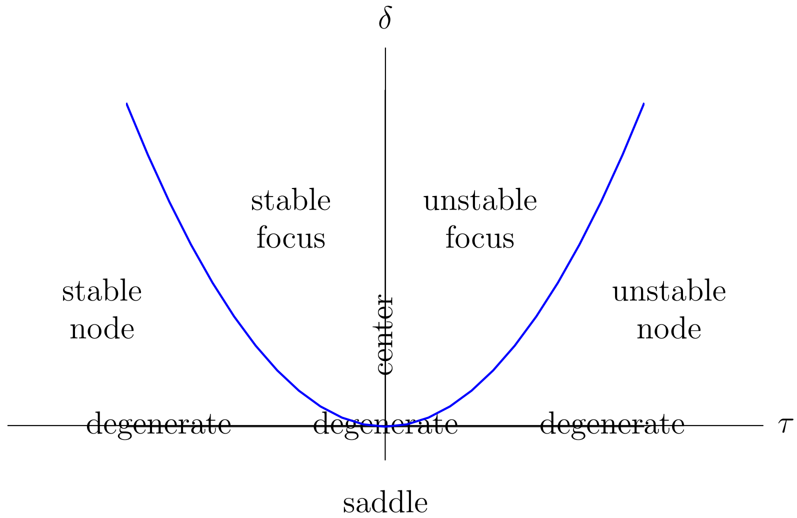
Proof. We classified possible nongenerate critical points earlier in the semester. The above corollary rules out all possibilities except for those listed above. In detail, the linearization of

$$\begin{aligned} x' &= H_y \\ y' &= -H_x. \end{aligned}$$

is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} H_{yx} & H_{yy} \\ -H_{xx} & -H_{xy} \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The trace of A, i.e., the sum of its eigenvalues is $\tau = \operatorname{tr}(A) = 0$, and the determinant of A, i.e., the product of its eigenvalues is $\delta = \det(A) = H_{xx}H_{yy} - H_{yx}^2$. Recall our earlier analysis of linear systems in $\mathbb{R}^2$:



In our case, $\tau = 0$, and we see that if $\delta(A) < 0$, the linearized system is a saddle. By Hartman-Grobman, the critical point in the original system is then a (topological) saddle. In the case $\det(A) > 0$, the linearized system is a center. As presented earlier in the course, that means that the critical point in the original system is either a center or a focus. However, the corollary to conservation of energy, proved above, rules out the latter case: by the second derivative test, if $\det(A) > 0$, then H has a strict local maximum or minimum. So the critical point cannot be a focus. □

Newtonian system with one degree of freedom. Consider the equation

$$x'' = f(x)$$

where $f \in C^1(I)$ for some open interval I. We can think of x'' as the acceleration of a particle of mass 1 moving along a line under a force given by f. We can change this into a planar first-order system with the substitution $y = x'$:

$$\begin{aligned} x' &= y \\ y' &= f(x). \end{aligned}$$

To see that this is a Hamiltonian system, we need to find a function $H(x, y)$ such that $H_y = y$ and $H_x = -f(x)$. Integrating the first equation with respect to y gives

$$H(x, y) = \frac{1}{2}y^2 + U(x),$$

for some function U. Taking the partial with respect to x then gives

$$H_x(x, y) = \frac{d}{dx}U(x) = -f(x),$$

and hence,

$$U(x) = -\int_{x_0}^{x} f(s)\, ds.$$

We call

$$T(y) := \frac{1}{2}y^2 = \frac{1}{2}\left(x'\right)^2$$

the *kinetic energy* and $U(x)$ the *potential energy*, and we see the total energy is the sum of the two:

$$H(x, y) = T(y) + U(x).$$

Theorem 3. The critical points of this Newtonian system lie on the x-axis. The point $(x_0, 0)$ is a critical point iff x_0 is a critical point of the function $U(x)$, i.e., iff $U'(x_0) = 0$. Suppose that H is analytic. Then,

1. If x_0 is a strict local maximum for U, then $(x_0, 0)$ is a saddle for the system.
2. If x_0 is a strict local minimum for U, then $(x_0, 0)$ is a center for the system.
3. If x_0 is a horizontal inflection point for U (which means its first nonzero derivative at x_0 of positive order is of an odd order), then $(x_0, 0)$ is a cusp (i.e., two hyperbolic sectors and two separatrices).

Proof. Exercise. □

Example. Consider the case of the undamped pendulum:

$$x'' = -\sin(x).$$

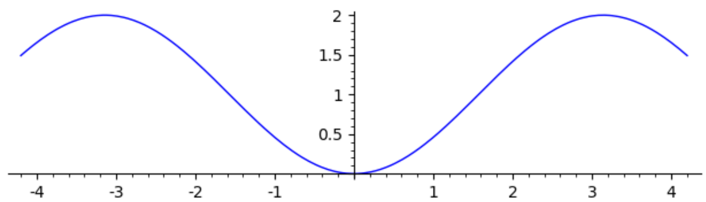
The corresponding first-order planar system is

$$\begin{aligned} x' &= y \\ y' &= -\sin(x). \end{aligned}$$

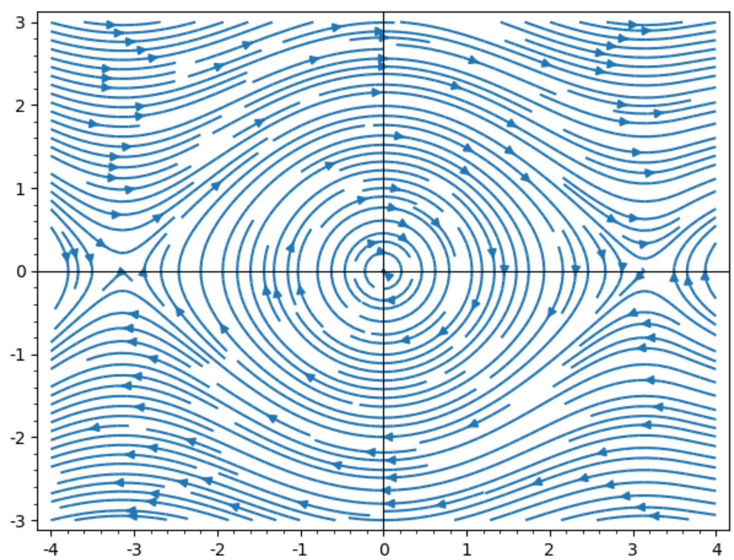
The potential energy function is

$$U(x) = \int_{0}^{x} \sin(s)\, ds = 1 - \cos(x).$$

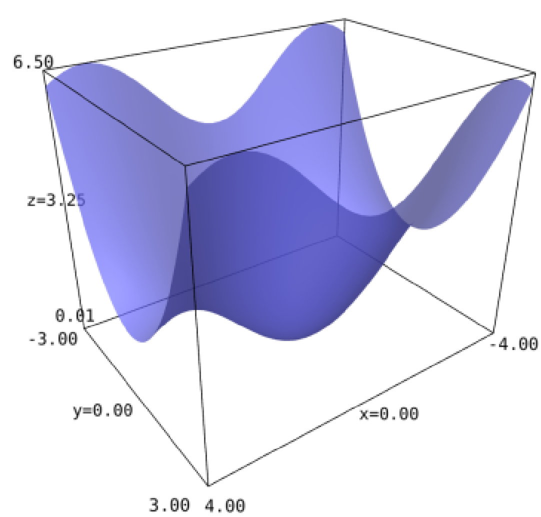
On the next page, we have pictures of both the potential energy and the phase portrait. Try to see how they reflect Theorem 3. Also note the physical meaning of the phase portrait. The x-axis shows the motion of the pendulum. The y-coordinate gives the velocity. The critical points are $n\pi$ for $n \in \mathbb{Z}$ and occur when the pendulum is balanced vertically or hanging straight down. If the velocity is high enough, the pendulum is continuously spinning around in a circle.



Graph of $U(x)$.



Phase portrait.



Graph of H.

Week 13, Monday: Gradient systems

GRADIENT SYSTEMS

Let $E \subseteq \mathbb{R}^n$ be an open subset, and let $V\colon E \to \mathbb{R}$ be a function in $C^2(E)$. The system

$$x' = -\operatorname{grad} V(x) = -\nabla V(x)$$

is called a *gradient system*.

The critical points of a gradient system occur where $\nabla V(x) = 0$, i.e., where are the partials of V vanish. Hence, they are exactly the critical points of the real-valued function V. A point that is not a critical point of V is called a *regular point* of V.

Example. Let

$$V(x, y) = x^2(x-1)^2 + y^2.$$

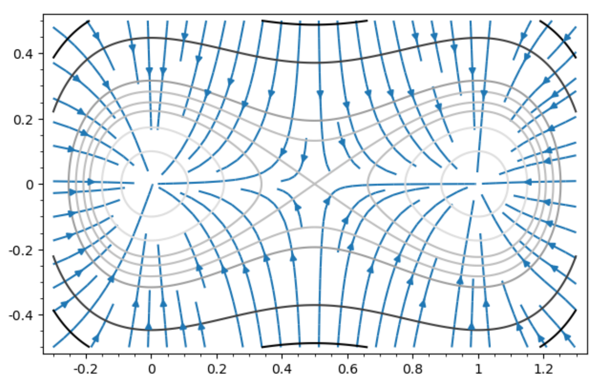
The corresponding gradient system is

$$\begin{aligned} x' &= -V_x = -2x(x-1)(2x-1) \\ y' &= -V_y = -2y. \end{aligned}$$

The critical points are

$$(0,0), (1/2,0), (1,0).$$

Here is a picture of the flow with a contour diagram for V superimposed:



Proposition. At a regular point p of V, any trajectory of the gradient system passing through p is perpendicular to the level set $V(x) = V(p)$. If q is a strict local minimum of V, then it is a stable equilibrium point for the gradient system; if, in addition, q is an isolated equilibrium point, then it is asymptotically stable.

Proof. Suppose that $\gamma(t)$ is a solution curve of the gradient system and that $\gamma(t_0) = p$. We need to show that $\gamma'(t_0)$ is perpendicular to the level set $V(x) = V(p)$. So take any curve $\lambda(t)$ sitting in the level set and such that $\lambda(0) = p$. Since it sits in the level set, we have $V(\lambda(t)) = V(p)$ for all t. It follows that

$$0 = \frac{d}{dt}V(\lambda(t)) = \nabla V(\lambda(t)) \cdot \lambda'(t).$$

Evaluating at $t = 0$, we get

$$0 = \nabla V(p) \cdot \lambda'(0) = -\gamma(t_0) \cdot \lambda'(0),$$

and hence, $\lambda'(0)$ is perpendicular to $\gamma'(t_0)$ at p.

Next, suppose that $q \in E$ is a strict local minimum of V. In particular, this implies that all of the partials of V vanish at q, and hence, q is a critical point of the system. Define $\widetilde{V}(x) := V(x) - V(q)$. Then in an open neighborhood U of q, the function $\widetilde{V}$ is positive except at q, where it is 0. Thus, $\widetilde{V}$ is a Liapunov function for the system in a neighborhood of V.

Let $\gamma(t)$ be a solution trajectory in U. Then,

$$\begin{aligned}
\dot{\widetilde{V}}(\gamma(t)) &= \frac{d}{dt}\widetilde{V}(\gamma(t)) \\
&= \nabla\widetilde{V}(\gamma(t)) \cdot \gamma'(t) \\
&= \nabla V(\gamma(t)) \cdot \gamma'(t) \\
&= -\nabla V(\gamma(t)) \cdot \nabla V(\gamma(t)) \\
&\leq 0
\end{aligned}$$

with equality if and only if $\nabla V(\gamma(t)) = 0$. From our earlier discussion of Liapunov functions, we see that q is a stable equilibrium point. If q is an isolated equilibrium point, then there exists a neighborhood of q on which $\nabla V \neq 0$ and hence a neighborhood on which $\dot{\widetilde{V}} < 0$. It follows that q is an asymptotically stable equilibrium point. $\square$

Cautionary example. Let $V\colon \mathbb{R} \to \mathbb{R}$ be given by

$$V(x) = \begin{cases} \int_0^{|x|} e^{-1/s^2}\left(1+\sin\left(1/s^2\right)\right)\, ds & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Then V is C^∞ but $\nabla V = V'$ has a zero besides the origin in any open neighborhood of the origin. Hence, the origin is a strict local minimum for V but is not an asymptotically stable equilibrium point for the gradient system corresponding to V.

Duality between planar Hamiltonian and gradient systems. Consider the following two systems:

$$\begin{aligned} x' &= P(x,y) \\ y' &= Q(x,y) \end{aligned} \tag{36.1}$$

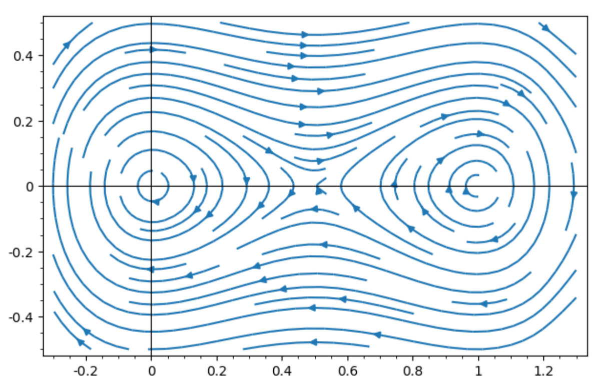
$$\begin{aligned} x' &= Q(x,y) \\ y' &= -P(x,y). \end{aligned} \tag{36.2}$$

By construction, the flows are perpendicular since $(P,Q)\cdot(Q,-P) = 0$. The critical points are the same. So centers of one correspond to nodes for the other, saddles of one correspond to saddles of the other, and foci correspond to foci. If one is a Hamiltonian system, then the other is a gradient system. For instance, suppose (36.1) is Hamiltonian, then there exists $H(x,y)$ such that $H_y = P$ and $H_x = -Q$. Then (36.2) is a gradient system with $V(x,y) = H(x,y)$.

Example. In the previous example, we considered the gradient system with $V(x,y) = x^2(x-1)^2 + y^2$. The corresponding dual system with Hamiltonian $H(x,y) = V(x,y)$ is

$$\begin{aligned} x' &= H_y = 2y \\ y' &= -H_x = -2x(x-1)(2x-1). \end{aligned}$$

The flow of this system appears below:



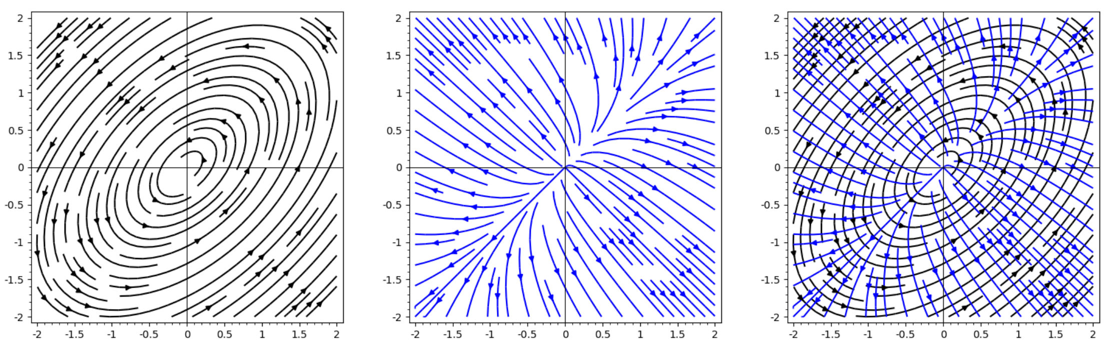
Example. Let $H(x,y) = V(x,y) = -x^2 + xy - y^2$, and consider the Hamiltonian system

$$\begin{aligned} x' &= P(x,y) = H_y = x - 2y \\ y' &= Q(x,y) = -H_x = 2x - y. \end{aligned} \tag{36.3}$$

The dual gradient system is

$$\begin{aligned} x' &= Q(x,y) = -V_x = 2x - y \\ y' &= -P(x,y) = -V_y = -x + 2y. \end{aligned} \tag{36.4}$$

Here are the flow diagrams for systems (36.3) and (36.4), in order and then combined:



Using duality, the following theorem is a consequence of the corresponding theorem for Hamiltonian systems given in the previous lecture.

Theorem. Let p be a nondegenerate critical point for a gradient system in $\mathbb{R}^2$ for which V is analytic. Then p is either a saddle or a node. If p is a saddle for V then it's a saddle for the system, and if p is a strict local minimum (resp. maximum) of V, then its a stable (resp. unstable) node for the system.