Math 322 Homework 1

The problems for the first assignment begin on page 3. Before beginning, please review the following expectations for homework from our Course information sheet:

Solutions. Excellent homework solutions take many forms, but they all have the following characteristics:

- they use complete sentences, even when formulas or symbols are involved;
- they are written as explanations for other students in the course; in particular, they fully explain all of their reasoning and do not assume that the reader will fill in details;
- when graphical reasoning is called for, they include large, carefully drawn and labeled diagrams;
- they are neatly typeset using the LaTeX document preparation system. A guide to LaTeX resources is available on the course homepage.

Recommendations. Here are some strategies for efficiently learning from the homework assignments:

- start early, don't wait until the night before it's due to look at it;
- read all the problems, and identify the ones you can solve right away and the ones you can't;
- review your notes and the book carefully; even if you paid attention in lecture, you probably didn't get all the details (I recommend doing this *before* trying to attempt the problems);
- make an honest attempt to solve all the problems before seeking help;
- talk to others, you can really learn from each other (make sure you don't just get the solutions from someone else, and that you are learning and understanding from this process);
- if needed, please come to my office hours. If the posted hours are not convenient for you, then please let me know and we will make other arrangements.

Feedback. You will receive timely feedback from me on your homework via Gradescope. Most homework problems will be graded on a five-point scale (5 = perfect; 4 =

minor mistake; 3 = major mistake, right idea; 2 = significant idea; 1 = attempted, 0 = none of the above). *The quality of your writing will be taken into account.* If your answer is incorrect, this will be reflected in the score, and there will also be a comment indicating where things went wrong with your solution. You are strongly encouraged to engage with this comment, understand your error, and try to come up with a correct solution. You are welcome to post questions about homework problems (old and new) to our Moodle forum and talk about them with me in office hours (see the Help section).

I reserve the right to not accept late homework. If health or family matters might impede the timely completion of your homework, please contact me as early as possible.

Collaboration. You are permitted and encouraged to work with your peers on homework problems. It is best practice to cite those with whom you worked, and you must write up solutions independently. **Duplicated solutions will not be accepted and constitute a violation of the Honor Principle.**

Gradescope. Submit your solutions document as a pdf (not an image file) to Gradescope, remembering to assign each problem to page(s) in your pdf. Overleaf templates will be provided for solutions, and I encourage you to use those.

Math 322 Homework 1

Solve the following differential equations using the methods from class. You can check your solutions with a computer, but what you turn in should be done by hand.

For each problem:

(i) give the solution to the equation satisfying the given initial condition $y(t_0) = I$, and

(ii) specify the largest open interval about t_0 in which your solution is valid.

PROBLEM 1. $y' = y^3$ with $y(0) = -2$.

Note: Do not leave y defined implicitly; solve for y in an interval about $t = 0$.

PROBLEM 2. $y' = y \sin(t)$ with $y(0) = 1$.

PROBLEM 3. $y' = \dfrac{3y - 2t}{t}$ with $y(1) = 4$.

Math 322 Homework 2

Solve each of the following differential equations. Your solution should have the form $y =$ etc. In other words, I'm looking for an explicit solution. Don't worry about the maximal interval in which your solution is defined, but if your initial condition is given at time t_0, make sure your solution is defined about the point t_0. Don't leave answers with complex numbers, e.g., use sines and cosines rather than e^{it}.

1. $y' = \dfrac{\cos t}{y}, \quad y(0) = -4.$
2. $2ty\, y' = t^2 + y^2, \quad y(1) = 0.$
3. $y' = y^2 + 2y + 1, \quad y(0) = -1.$
4. $3t^2 y + y + (t^3 + t + 2y)y' = 0, \quad y(0) = 2.$
5. $e^{-t} y' = 3e^{-t} y + 1, \quad y(0) = 0.$
6. $y' + y = ty^3, \quad y(0) = 1.$
7. $y'' - y' - 12y = 0, \quad y(0) = 1, \quad y'(0) = 2.$
8. $y'' + 25y = 0, \quad y(0) = 1, \quad y'(0) = -1.$
9. $8y'' + 2y' - y = 0, \quad y(-1) = 1, \quad y'(-1) = -2.$
10. $y''' - 6y'' + 9y' = 0, \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = -1.$

Math 322 Homework 3

PROBLEM 1. Solve each of the following differential equations. Your solution should have the form $y =$ etc. In other words, I'm looking for an explicit solution. Don't leave answers with complex numbers, e.g., use sines and cosines rather than e^{it}.

1. $y'' - 2y' + y = 2\cos(t) + 4e^{3t}, \quad y(0) = y'(0) = 1.$
2. $ty' + 5y - t^5y^2 = 0, \quad y(1) = 1.$
3. $y'' + 2y' + 3y = 5 + 3t.$
4. $y' = yt/(t^2 + 1), \quad y(0) = -3.$
5. $y'' - 6yy' = 0, \quad y(0) = 2,\ y'(0) = 9.$
6. $y'' - 6yy' = 0, \quad y(0) = 2,\ y'(0) = 0.$
7. $D^2(D + 1)^3(D^2 + 2D + 2)^2y = 0.$
8. $y^{(4)} - 16y = 0.$
9. $y'' = -2(y')^2, \quad y(0) = 1 = y'(0) = 1.$

PROBLEM 2. Let $A \in M_n(F)$, and let r_i be the i-th row of A for $i = 1 \ldots n$. Let $\ell = \max\{|r_i| : 1 \leq i \leq n\}$, the maximum length of a row of A. Prove $\|A\| \leq \ell\sqrt{n}$. (Strive to find an elegant solution that does not involve referencing the elements of A by name. You can do everything using just the notation introduced in the statement of the problem. Consider $|Ax|^2$ for $|x| \leq 1$, and use Cauchy-Schwarz. On the other hand, an ugly solution is still a solution.)

Math 322 Homework 4

Problem 1. (Product rule). Let $M_{m\times n}(F)$ denote the space of $m \times n$ matrices over $F = \mathbb{R}$ or $\mathbb{C}$. The derivative of a function

$$\begin{aligned} F &\to M_{m\times n}(F) \\ t &\mapsto A(t) \end{aligned}$$

with respect to t is defined entrywise:

$$(A(t)')_{ij} = (A(t)_{ij})'.$$

Let $A(t)$ and $B(t)$ be two such functions into $M_{m\times p}(F)$ and $M_{p\times n}$, respectively. Use the ordinary product rule from one-variable calculus to prove the product rule

$$(A(t)B(t))' = A(t)'B(t) + A(t)B(t)'.$$

(Use the definition of multiplication of matrices using summation notation. Do not write out matrices with ellipses.)

Problem 2. Consider the system

$$\begin{aligned} x_1' &= x_1 + 4x_2 \\ x_2' &= 4x_1 + x_2 \end{aligned}$$

Find the solution to this system with initial condition $x(0) = (1, 3)$ by diagonalizing a matrix and exponentiating by hand. Don't use a computer (except to check your work, if you'd like), and show your work.

Problem 3. Let $A \in M_n(F)$ and let $W \subseteq F^n$ be a subspace. Suppose W is invariant under A, i.e., $Aw \in W$ for all $w \in W$. Let $x' = Ax$ have solution $x(t)$ with $x(0) = x_0 \in W$. The goal of this problem is to show that $x(t)$ never leaves the subspace W. To prove this, fix t and define the sequence

$$x_n = \left(\sum_{k=0}^{n} \frac{A^k t^k}{k!} \right) x_0$$

for each $n \geq 0$. Since $Ax_0 \in W$, it easily follows that $x_n \in W$ for all n.

Now, the space W is complete, i.e., every Cauchy sequence in W converges to a point in $w \in W$. That's because W is linearly isomorphic to F^m where $m = \dim W$, which is complete.) Therefore, if we can show that (x_n) is a Cauchy sequence, the result will follow since

$$x(t) = e^{At}x_0 = \lim_{n\to\infty} x_n = w \in W.$$

Problem. Your job is to prove that the sequence (x_n) is a Cauchy sequence. You may use the fact that e^{At} is Cauchy for each t (as shown in class). Give an ε-N proof. Lemma 1, from the lecture on Monday Week 3 may be of use.

PROBLEM 4. (D'Alembert reduction trick.) Suppose you have found a solution $a(t)$ to a differential equation of degree n

$$q_n(t)y^{(n)} + q_{n-1}(t)y^{(n-1)} + \cdots + q_1(t)y' + q_0(t)y = 0.$$

Substitute $y(t) = u(t)a(t)$. Then, after cancellation, substitute $v = u'$ to get an equation of the form

$$aq_n(t)v^{(n-1)} + p_{n-2}(t)v^{(n-2)} + \cdots + p_0(t)v = 0,$$

of degree $n-1$.

(a) Write out the details of this reduction procedure for the case $n = 3$.

(b) Apply the reduction procedure to find the most general solution to

$$t^2y'' - 3ty' + 4y = 0$$

given the solution $a(t) = t^2$. You may assume that the initial conditions are such that $t > 0$ and the substituted quantities are positive.

Math 322 Homework 5

Problem 1. Recall that for the Jordan block matrix $J_k(\lambda)$, we have

$$(J_k(\lambda) - \lambda I_k)e_1 = e_1,$$

i.e., e_1 is an eigenvector for $J_k(\lambda)$, and

$$(J_k(\lambda) - \lambda I_k)e_i = e_{i-1}$$

for $i = 2, \ldots, k$. So in order to put an $n \times n$ matrix A into Jordan form, for each eigenvalue λ, we look for vectors v_i such that

$$(A - \lambda I_n)v_1 = v_1,$$

and

$$(A - \lambda I_n)v_i = v_{i-1}$$

for $i = 2, \ldots$ These v_i will end up as columns in a matrix P for which $P^{-1}AP$ is in Jordan from.

This problem will consider a simple example of this procedure. Let

$$A = \begin{pmatrix} 1 & 4 \\ -1 & 5 \end{pmatrix}.$$

Do all of the following exercises by hand, and show your work.

(a) Compute the characteristic polynomial $p(x)$ for A, and factor it to find the single eigenvalue λ for A (with multiplicity 2).

(b) Find an eigenvector v_1 for λ with integer components (to make things simple).

(c) Find a vector v_2, again with integer components, such that

$$(A - \lambda I_2)v_2 = v_1.$$

(d) Let P be the matrix with columns v_1 and v_2 (in that order), and show that $P^{-1}AP$ is in Jordan form.

Problem 2. Find all possible Jordan forms for a matrix with a single real eigenvalue $u \in \mathbb{R}$ of multiplicity 4 (up to permutations of the blocks).

Problem 3. Let $A \in M_4(\mathbb{R})$ with two *not necessarily distinct* real eigenvalues u and v and one pair of conjugate non-real eigenvalues $a \pm bi$. Find the possible real Jordan forms for A (up to permutation of blocks and $\pm b$).

Problem 4. Describe all possible Jordan forms for a real 2×2 degenerate system (i.e., with determinant 0). There are two possibilities for which 0 is a repeated eigenvalue and an infinite class of possibilities for which 0 is an eigenvalue of multiplicity 1. For each, describe the solution to $x' = Jx$ with initial condition $x_0 = (\alpha, \beta) \in \mathbb{R}^2$.

Problem 5. Solve the system

$$x' = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

with initial condition $x(0) = (7, 5) \in \mathbb{R}^2$, using the method from section 1.10 of our text. (Using the notation from section 1.10, take $\Phi(t) = e^{At}$. Show your work).

Math 322 Homework 6

Problem 1. Let $A \in M_n(F)$ and let $t \mapsto b(t) \in F^n$ be continuous. Let $x_0 \in F^n$, and let u and v be solutions to the initial value problem

$$x'(t) = Ax(t) + b(t) \quad \text{and} \quad x(0) = x_0. \tag{1}$$

We have shown that for each initial condition $y_0 \in F^n$, that the solution to $y'(t) = Ay(t)$ with $y(0) = y_0$ is unique. Use this result to show that $u = v$ (i.e., the solution to system (1) is unique.)

Problem 2. We found that the solution to the forced harmonic oscillator problem

$$x'' = -x + f(t)$$

has the solution

$$x(t) = x(0)\cos(t) + x'(0)\sin(t) + \int_{s=0}^{t} f(s)\sin(t-s)\,ds.$$

We also saw by integrating that in the case $f(t) = \cos(\omega t)$, the solution is

$$x(0)\cos(t) + x'(0)\sin(t) + \frac{\cos(\omega t) - \cos(t)}{1-\omega^2}.$$

While solving this equation, at some point we assumed $\omega \neq \pm 1$.

(a) Go back to our solution and revise it to get a solution in the case where $\omega = 1$, and thus solve the forced harmonic oscillator problem with $f(t) = \cos(t)$. Use the identity

$$\sin(\theta + \psi) + \sin(\theta - \psi) = 2\cos(\psi)\sin(\theta),$$

and show your work.

(b) Graph the solution with initial condition $x(0) = x'(0) = 1$, enough to get a qualitative sense of the nature of the solution.

Problem 3. Consider the n-th order differential equation with constant coefficients

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

The characteristic polynomial for the equation is $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. Define the differential operator $D = \frac{d}{dt}$. Then our differential equation may be written

$$P(D)y = 0.$$

Suppose that P factors as $P(x) = \prod_{j=1}^{k}(x-\lambda_j)^{m_j}$ where the λ_j are distinct. We would like to show that the *basic functions* for our equation,

$$\left\{t^j e^{\lambda_i t} : 0 \le j \le m_i - 1,\ 1 \le i \le k\right\}$$

are solutions. So we need to show for each i that

$$P(D)(t^\ell e^{\lambda_i t}) = 0 \tag{2}$$

for $0 \le \ell \le m_i - 1$. We do this in steps.

(a) Prove by induction that for every sufficiently differentiable function $f(t)$, we have

$$(D-\lambda)^k(f(t)e^{\lambda t}) = e^{\lambda t} D^k f(t)$$

for $k \ge 0$.

(b) Use the above result to prove that for each i, equation (2) holds for $0 \le \ell \le m_i - 1$. You may use the fact that since D commutes with constants and with itself,

$$(D-\lambda)(D-\mu) = (D-\mu)(D-\lambda).$$

Problem 4. Let $f(t)$ be a real-valued integrable function on some open interval I containing 0, and let $x_0 \in \mathbb{R}$. Consider the initial value problem

$$\begin{aligned} x'(t) &= f(x(t)) \\ x(0) &= x_0. \end{aligned}$$

By the fundamental theorem of calculus,

$$x(t) := x_0 + \int_{s=0}^{t} f(x(s))\, ds$$

is a solution. (You could check by computing $x'(t)$ and $x(0)$.) Even if we cannot compute the integral directly, we can attempt to find a solution via the method of *successive approximations*. Define

$$u_0(t) := x_0$$

and for $k \ge 0$,

$$u_{k+1}(t) := x_0 + \int_{s=0}^{t} f(u_k(s)).$$

Consider the case where $f(t) = \lambda t$ for some $\lambda \in \mathbb{R}$.

(a) Apply the method of successive approximations to find u_1, u_2, and u_3.

(b) Identify $\lim_{n\to\infty} u_n$. No proof is necessary.

(c) Solve the initial value problem exactly using methods we already know. (Your solution should agree with the limit you just calculated.)

Math 322 Homework 7

Problem 1. Let X, Y be subsets of normed linear spaces $(V, \| \|_V)$ and $(W, \| \|_W)$, respectively, and suppose $f\colon X \to Y$. Then f is *continuous* if for all $u \in X$ and for all $\varepsilon > 0$, there exists $\delta = \delta(u, \varepsilon) > 0$ such that $\|u - v\|_V < \delta$ implies $\|f(u) - f(v)\|_W < \varepsilon$.

(a) For any normed linear space $(V, \| \|)$, prove that $\| \|\colon V \to \mathbb{R}$ is continuous (with the usual norm on $\mathbb{R}$).

(b) Let X be a subset of a normed linear space $(V, \| \|)$. Suppose $T\colon X \to X$ is a contraction mapping. Prove that T is continuous.

Problem 2. Our existence-uniqueness theorem applies to an initial value problem

$$\begin{aligned} x' &= f(x) \\ x(0) &= x_0 \end{aligned}$$

where f is a continuously differentiable function. If f is just continuous, it no longer applies. Here is an example: consider the initial value problem

$$\begin{aligned} x' &= 2\sqrt{x} \\ x(0) &= 0. \end{aligned}$$

For each $a > 0$, define

$$x_a(t) := \begin{cases} 0 & \text{if } t \leq a \\ (t-a)^2 & \text{if } t > a. \end{cases}$$

(a) Sketch the graph of $x_a(t)$.

(b) Each $x_a(t)$ is clearly differentiable away from $t = a$. Use the definition of the derivative to prove that $x_a(t)$ is differentiable at $t = a$.

(c) Show that each $x_a(t)$ solves the initial value problem.

Problem 3. Read Theorems 1 and 2 in Section 2.4 (*The Maximal Interval of Existence*). Consider the initial value problem

$$\begin{aligned} x_1' &= x_1^2 & \qquad x_1(0) &= 1 \\ x_2' &= x_2 + \frac{1}{x_1} & \qquad x_2(0) &= 1. \end{aligned}$$

(a) Solve the initial value problem showing your technique.

(b) What is the maximal interval of existence (α, β)?

(c) Use a computer to draw the vector field and your solution in a single plot.

(d) How is Theorem 2 exemplified by your solution?

(e) What is the speed of your solution at each time t in the interval of existence?

Math 322 Homework 8

Problem 1. We recently considered linearizing a system $x' = f(x)$ at an equilibrium point x_0, i.e., at a point where $f(x_0) = 0$ by considering the system $x' = Jf(x_0)x$ where $Jf(x_0)$ is the Jacobian matrix of f as the origin. We could hope that the linearized system would determine the nature of the equilibrium point for the original system. The following example shows that hope is unfounded if the linearized system has a center at the origin. (It turns out that in general for this case, x_0 is either a center or a focus for the original system.) Consider the system

$$\begin{aligned} x' &= -y + xy^2 \\ y' &= x + y^3 \end{aligned}$$

(a) What is the linearized system at the equilibrium point $(0, 0)$?

(b) What are the eigenvalues for the linearized system? (You'll find the real parts of these are 0, and hence describe a center.)

(c) Convert to polar coordinates: $r^2 := x^2 + y^2$, $x = r\cos(\theta)$, and $y = r\sin(\theta)$. For the original system, prove that $r' = r^3 \sin^2(\theta)$ and $\theta' = 1$.

(d) What does part (c) say about solutions for our original system?

(e) Use a computer to plot the vector field for the original system. Here is sample code for Sage:

```
x,y = var('a,b')
plot_vector_field((-y,x),(x,-2,2),(y,-2,2),aspect_ratio=1)

    or

streamline_plot((-y,x),(x,-2,2),(y,-2,2),aspect_ratio=1,density=2)
```

(Please see the posting for last week's homework on our class homepage for instructions on including a plot in your LaTeX document.)

Problem 2. Consider the system

$$\begin{aligned} x' &= x - xy \\ y' &= y - x^2 \end{aligned}$$

(a) Find all the equilibrium points x_0, i.e, the points x_0 at which $f(x_0) = 0$.

(b) For each equilibrium point x_0, compute the Jacobian $Jf(x_0)x$ and consider the linear system

$$\begin{aligned} x' &= Jf(x_0)x \\ x(0) &= x_0 \end{aligned}$$

Classify the origin for this linearized system as a saddle, a stable or unstable node, a stable or unstable focus, or a center.

(c) Use a computer to plot the vector field, including all the equilibrium points.

Projective space. Here is an example of an important manifold, n-dimensional projective space, $\mathbb{P}^n$. As a set, $\mathbb{P}^n$ is the collection of one-dimensional linear subspaces of $\mathbb{R}^{n+1}$, i.e., all lines through the origin in $\mathbb{R}^{n+1}$. Every one-dimensional subspace is the same thing as the span of some nonzero vector. Two nonzero vectors $x, y \in \mathbb{R}^{n+1}$ determine the same one-dimensional subspace if and only if there is a nonzero scalar λ such that $x = \lambda y$, and in this case we will write $x \sim y$. Then $\sim$ is an equivalence relation on nonzero elements of $\mathbb{R}^{n+1}$. The equivalence classes are in one-to-one correspondence with one-dimensional subspaces and hence with points in $\mathbb{P}^n$. Therefore, sometimes one will see the following definition for projective space

$$\mathbb{P}^n = \left(\mathbb{R}^{n+1} \setminus \{0\}\right) / \left(x \sim \lambda x,\ \lambda \neq 0\right).$$

Abusing notation, one usually refers to a point in $\mathbb{P}^n$ as $x = (x_0, \ldots, x_n)$ when one really means the one-dimensional space spanned by x. In that case, $x_0, \ldots, x_n$ are called the *homogeneous coordinates* of the point in projective space.

Recall that a manifold is a connected metric space M with an *atlas*. The atlas is a collection of pairs (h_α, U_α) where each U_α is an open subset of M and h_α is a homeorphism of U_α to some open subset V_α of $\mathbb{R}^n$. We require that the union of the U_α is M and if $U_\alpha \cap U_\beta \neq \emptyset$, then the *transition function*

$$h_\beta \circ h_\alpha^{-1} : h_\alpha(U_\alpha \cap U_\beta) \to h_\beta(U_\alpha \cap U_\beta)$$

is a differentiable function. The standard atlas for $\mathbb{P}^n$ consists of $n+1$ charts (h_i, U_i) where, for $i = 0, \ldots, n$,

$$U_i = \{(x_0, \ldots, x_n) \in \mathbb{P}^n : x_i \neq 0\}$$

and

$$\begin{array}{rccc} h_i \colon & U_i & \to & \mathbb{R}^n \\ & (x_0, \ldots, x_n) & \mapsto & \left(\dfrac{x_0}{x_i}, \ldots, \dfrac{\widehat{x_i}}{x_i}, \ldots, \dfrac{x_n}{x_i}\right) \end{array}$$

where the symbol $\widehat{\ }$ is used to denote omitting the i-th term. Also note that U_i is well-defined since if $x_i \neq 0$ then $\lambda x_i \neq 0$ for every $\lambda \neq 0$.

For the following exercises, we let $n = 2$ and consider the projective plane, $\mathbb{P}^2$.

(a) Explicitly describe $h_i(x_0, x_1, x_2)$ for $i = 0, 1, 2$.

(b) Compute the inverse of h_0, mapping $\mathbb{R}^2 \to U_0$.

(c) Compute the transition function $h_1 \circ h_0^{-1}$.

(d) Compute the Jacobian matrix for the transition function $h_1 \circ h_0^{-1}$, and explain why its entries are continuously differentiable for each $p \in h_0(U_0 \cap U_1)$? (Hence, this transition function is continuously differentiable, and by symmetry, so are all the others.)

Math 322 Homework 9

PROBLEM 1. Consider the system

$$\begin{aligned} x' &= -x \\ y' &= y + x^3. \end{aligned}$$

(a) Use the method of successive approximations given in class to compute the stable manifold for the equilibrium at the origin. (It converges fairly quickly.)

(b) Compute the exact solution with arbitrary initial condition (x_0, y_0) using methods from the beginning of the semester. (Any constants you use should be expressed in terms of x_0 and y_0.) Show your work.

(c) From your answer to part (b), find the initial conditions (x_0, y_0) such that corresponding solution converges to $(0,0)$ as $t \to \infty$. (Your answer to part (a) should provide a check!)

(d) From your answer to part (b), find the initial conditions (x_0, y_0) such that corresponding solution converges to $(0,0)$ as $t \to -\infty$, the unstable manifold.

(e) Draw the vector field with superimposed stable and unstable manifolds at $(0,0)$.

PROBLEM 2. Solve the system

$$\begin{aligned} x' &= -x \\ y' &= -y + x^2 \\ z' &= z + y^2 \end{aligned}$$

using methods from the beginning of the semester, and use your solution to show that the points in the stable manifold satisfy

$$z = -\frac{1}{3}y^2 - \frac{1}{6}x^2 y - \frac{1}{30}x^4$$

and the points on the unstable manifold satisfy

$$x = y = 0.$$

(It would be great if anyone could come up with a nice visualization of this!)

PROBLEM 3. **Sinks, sources, and saddles.** Consider the differential equation $x' = f(x)$. Any point x_0 in the domain of f such that $f(x_0) = 0$ is called an *equilibrium point* or *critical point* for the equation. An equilibrium point x_0 is called a *sink* if all eigenvalues of the Jacobian matrix $Jf(x_0)$ have negative real part; it is called a *source* if all eigenvalues of $Jf(x_0)$ have positive real part; and it is called a *saddle* if $Jf(x_0)$ has at least one eigenvalue with positive real part, at least one eigenvalue with negative real part, and no eigenvalues with zero real part. Equilibrium points like these, with no eigenvalue having real part equal to zero, are called *hyperbolic equilibrium points.* The Hartman-Grobman theorem says that near a hyperbolic equilibrium point x_0, a nonlinear system $x' = f(x)$ will behave qualitatively like the associated linear system $x' = Jf(x_0)x$ does near the origin.

For the systems with $f(x)$ as follows, (i) find all equilibrium points, and (ii) classify each hyperbolic equilibrium point as a sink, source, or saddle, and (iii) draw a picture of the vector field which includes all of the equilibrium points.

(a) $\begin{pmatrix} x_1 - x_1 x_2 \\ x_2 - x_1^2 \end{pmatrix}$

(b) $\begin{pmatrix} -4x_2 + 2x_1 x_2 - 8 \\ 4x_2^2 - x_1^2 \end{pmatrix}.$

Math 322 Homework 10

Problem 1. The equilibrium point at the origin for the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

is a center (since the eigenvalues of the linear function on the right are $\pm i$). In this problem, we show that perturbing the system a little can lead to various types of equilibrium points at the origin.

Establish the following results using the Liapunov function $V(x,y) = x^2 + y^2$:

(a) The origin is an asymptotically stable equilibrium point for the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -x^3 - xy^2 \\ -y^3 - x^2y \end{pmatrix}.$$

(b) The origin is an unstable equilibrium point for the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x^3 + xy^2 \\ y^3 + x^2y \end{pmatrix}.$$

(c) The origin is a stable equilibrium point which is not asymptotically stable for the system. What can you say about the solution trajectories in this case?

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -xy \\ x^2 \end{pmatrix}.$$

(d) Draw the vector field or the flow diagrams for the above three systems in order to verify your results.

Problem 2. The point of this problem is to give an example of an asymptotically stable equilibrium point that is not stable. Consider the following system, defined on $\mathbb{R}^2 \setminus \{(0,0)\}$:

$$x' = x - y - x(x^2 + y^2) + \frac{xy}{\sqrt{x^2 + y^2}}$$

$$y' = x + y - y(x^2 + y^2) - \frac{x^2}{\sqrt{x^2 + y^2}}.$$

(a) Converting to polar coordinates, $x = r\cos(\theta)$, $y = r\sin(\theta)$, since $r^2 = x^2 + y^2$, we have

$$rr' = xx' + yy',$$

which makes it easy to compute r' directly from the equation for the system. Similarly, it is easy to check (by substituting the polar coordinates for x and y) that

$$r^2\theta' = xy' - x'y.$$

For this problem, find an expression for r' completely in terms of r and an expression for θ' completely in terms of θ. You may have use for the half-angle formula

$$\sin^2(\theta/2) = \frac{1-\cos(\theta)}{2}.$$

(b) Solve the converted system completely, i.e., for each initial condition. Your solution should write r^2 as a function of time and the initial condition r_0 and should write $\cot(\theta/2)$ as a function of time and the initial condition θ_0. Show your work (you may need to review integration using partial fractions and integration of trig functions.) Don't forget the "duh" solutions where $\theta = 0$ or $r = 1$.

(c) Show that the equilibrium point $(1, 0)$ is asymptotically stable but not stable. (It might help to remember what the plot of the cotangent function looks like.)

(d) Create a phase portrait (picture of the flow of the vector field). (With Sage, you can use `streamline_plot`.)

Math 322 Homework 11

PROBLEM 1. Use the index formula

$$I_f(C) = \frac{1}{2\pi} \oint_C \frac{P\,dQ - Q\,dP}{P^2 + Q^2}$$

to compute the index of a saddle at the origin.

PROBLEM 2. Let $z = x + iy$ and consider the vector field in the complex plane given by

$$z' = x' + iy' = z^k$$

where $k \in \mathbb{Z}$. Thus, we are interested in the system

$$\begin{aligned} x' &= \mathrm{Re}(z^k) \\ y' &= \mathrm{Im}(z^k). \end{aligned}$$

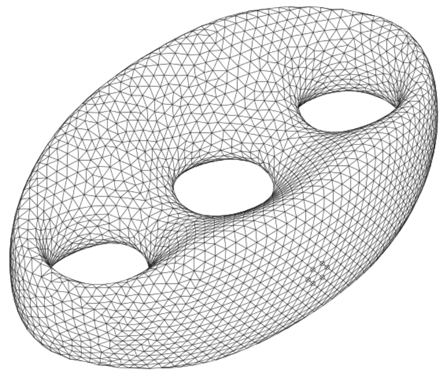
The origin is the unique critical point for the system.

(a) What is the index of the origin, in general, for the system $z' = z^k$? Explain. Hint: write $z = re^{i\theta}$ and imagine traveling counterclockwise around a unit circle centered at the origin. At the point $e^{i\theta}$ on the circle, what is the angle of the vector z^k? What is the total change as your go around the circle?

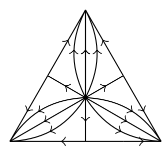
(b) Draw the vector field for the case $k = 3$.

(c) Draw the vector field for the case $k = -3$. (For a check, you could make sure for yourself that the index in these last two parts agrees with your answer to part (a).)

PROBLEM 3. Let M be a compact oriented two-dimensional manifold. It turns out that this means that M is a donut with g holes for some $g \in \mathbb{N}$. Triangulate M: draw triangles on the surface so that every point in M is in some triangle and if two triangles meet, they either do so vertex-to-vertex or edge-to-edge:



Let V be the number of vertices, E the number of edges, and F the number of faces of the triangulation. Create a vector field on each triangle vanishing at seven points, as shown below:



Consider the resulting vector field on M. What is the sum of the indices of the critical points in terms of V, E, and F? Explain.

Problem 4. Let $D := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ be the unit disk in the plane centered at the origin. Suppose that $\phi\colon D \to D$ is a smooth function. We would like to use index theory to prove that ϕ has a fixed point on D, i.e., there is a point $p \in D$ such that $\phi(p) = p$.

(a) Define a vector field f on D by

$$f(p) := \phi(p) - p.$$

Let C be the boundary of D, and suppose that f does not vanish anywhere on C. What is $I_f(C)$, the index of C relative to f? Explain.

(b) Explain why part (a) would lead to a contradiction if ϕ had no fixed points.

Math 322 Homework 12

Problem 1. For this problem please refer to the notes for Friday, Week 11, and Monday, Week 12. In the first part of these notes, we described how to induce a flow on the unit sphere centered at the origin in $\mathbb{R}^3$ and then analyze a critical point $(a, b, 0)$ on the equator by projecting the flow to the $x = 1$ plane. We gave coordinates u, v to the $x = 1$ plane and derived a system of differential equations in these coordinates. To analyze the critical point $(a, b, 0)$ on the equator, we could then analyze the critical point $\left(\frac{b}{a}, 0\right)$ in the u, v-plane. Antipodal points have similar behavior except that directions may be reversed depending on the parity of $d = \max\{\deg P, \deg Q\}$, using the notation in the handout.

Carry out the same analysis for projection to the $y = 1$ plane (this time assuming $b \neq 0$) to derive the system of equations

$$u' = v^d \left(P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right) \right)$$

$$v' = -v^{d+1} Q\left(\frac{u}{v}, \frac{1}{v}\right).$$

Explain your steps.

Problem 2. For each system below

i. Find and classify each critical point in the plane (sink, source, saddle, etc.)

ii. Determine and analyze the critical points at infinity (projecting to the $x = 1$ plane unless the critical point is $(0, \pm 1, 0)$, in which case, project to the $y = 1$ plane).

iii. Draw the global phase portrait. (For ease of TeX-ing, I would recommend using a tablet or hand-drawing the phase portrait and taking a photo. Then include the resulting filed using `\includegraphics`.)

iv. Use a computer to create a picture of the vector field or flow. (In Sage, you may want to use `streamline_plot` instead of `plot_vector_field`.)

(a)

$$x' = 2x$$
$$y' = y.$$

(b)

$$x' = x - y$$
$$y' = x + y.$$

(c)

$$x' = 2x - 2xy$$
$$y' = 2y - x^2 + y^2.$$

(d) In this problem, you'll get critical points at ∞ that aren't isolated. Just analyze the one at $(1, 0, 0)$.

$$x' = x$$
$$y' = y.$$