Math 322 Practice for Wednesday, Week 1

These are practice problems. They will not be collected, but solutions will be posted. In the following $y = y(t)$.

1. Solve the differential equation $y' = ty^2$ with the initial condition $y(0) = 3$.

SOLUTION:

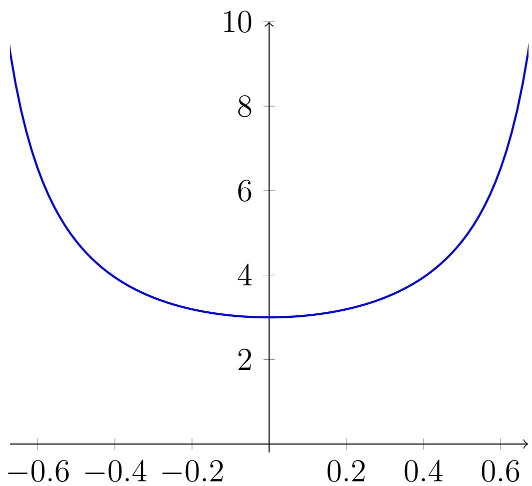
$$\begin{aligned} y' = ty^2 \quad &\Rightarrow \frac{y'}{y^2} = t \\ &\Rightarrow \int \frac{dy}{y^2} = \int t\,dt \\ &\Rightarrow -y^{-1} = \frac{1}{2}t^2 + c \\ &\Rightarrow y = -\frac{2}{t^2 + a}. \end{aligned}$$

For the initial condition, we have

$$3 = y(0) = -\frac{2}{a} \quad \Rightarrow \quad a = -\frac{2}{3}.$$

So the solution is:

$$y = -\frac{6}{3t^2 - 2}.$$



Graph of $y(t) = -6/(3t^2 - 2)$.

The largest interval about $t = 0$ for which the solution is defined is $(-\sqrt{2/3}, \sqrt{2/3})$.

2. Solve the differential equation $y' = 4te^{-y}$ with initial condition $y(0) = -1$.

SOLUTION:

$$\begin{aligned} y' = 4te^{-y} \quad &\Rightarrow \quad e^y y' = 4t \\ &\Rightarrow \quad \int e^y \, dy = \int 4t \, dt \\ &\Rightarrow \quad e^y = 2t^2 + c \\ &\Rightarrow \quad y = ae^{2t^2}. \end{aligned}$$

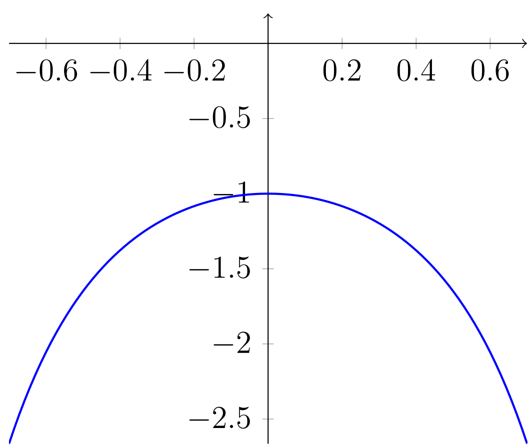
Initial condition:

$$-1 = y(0) = a.$$

The solution is:

$$y = -e^{2t^2}$$

The solution is defined for all $t \in \mathbb{R}$.



Graph of $y(t) = -e^{2t^2}$.

3. Consider the differential equation $y' = r(S - y)$ where r and S are positive constants. In the lecture notes for Monday, Week 1, we found that if we assume $y < S$, the solution is

$$y = S - (S - I)e^{-rt},$$

where $I = y(0)$.

(a) What is the solution if we assume $y > S$? (Express you solution as close to the solution for the $y < S$ case as you can.)

SOLUTION: Assume $y > S$. Then

$$\begin{aligned}\int \frac{dy}{S-y} = \int r\,dt \quad &\Rightarrow \quad -\ln|S-y| = rt + c \\ &\Rightarrow \quad |S-y| = ae^{-rt} \\ &\Rightarrow \quad y - S = ae^{-rt} \\ &\Rightarrow \quad y = S + ae^{-rt}.\end{aligned}$$

For the initial condition, we have

$$I := y(0) = S + a \quad \Rightarrow \quad a = -(S - I).$$

The solution is

$$y = S - (S - I)e^{-rt}.$$

(b) What is the solution if $y = S$?

SOLUTION: We get the constant solution: $y(t) = S$ for all t.