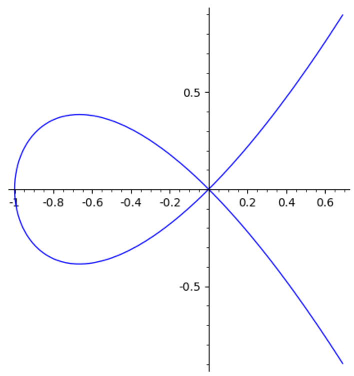
Math 322 lecture for Wednesday, Week 12

BLOWING UP CRITICAL POINTS

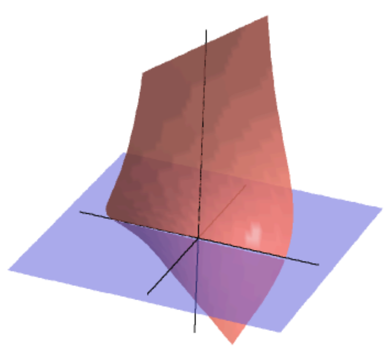
We would like to describe the process of *blowing up* a singularity in an algebraic curve. Consider the nodal cubic curve C

$$y^2 = x^3 + x^2 = x^2(x+1)$$

in the plane. By this we mean the set $\{(x,y) \in \mathbb{R}^2 : y^2 = x^3 + x^2\}$:



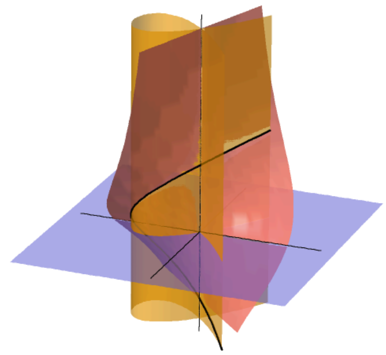
To blow up the singularity at $(0,0)$, we embed the curve in the $z = 0$ plane in $\mathbb{R}^3$. Next consider the "corkscrew" surface B in $\mathbb{R}^3$ defined by the equation $y = zx$. Intersecting this surface with the plane $z = m$ for each constant m, we get the line $y = mx$ with slope m:



Define the cylinder over the curve C:

$$\mathrm{Cyl}(C) := \left\{(x,y,z) \in \mathbb{R}^3 : (x,y) \in C\right\},$$

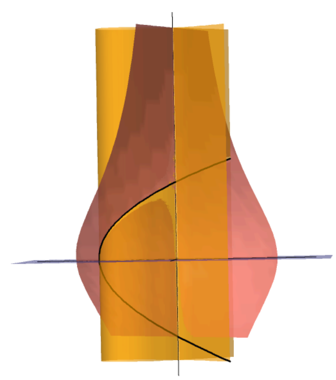
and imagine the intersection $\mathrm{Cyl}(C) \cap B$ of this cylinder with the corkscrew surface:



Algebraically, the intersection is the set of solutions (x, y, z) to the equations

$$y^2 = x^3 + x^2 \quad \text{and} \quad y = zx.$$

Substituting $y = zx$ into the first equation. We see that either the solution is the z-axis, and the curve on the surface $y = zx$ satisfying $z^2 = x + 1$. The z-axis is called the *exceptional divisor* and corresponds to the singularity of C. Project the curve on the corkscrew surface to the $y = 0$ plane to get the curve $z^2 = x + 1$, a parabola:

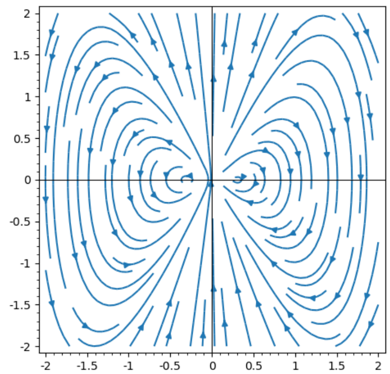


This parabola is the blow up of the nodal cubic at the origin. In intersection of the cylinder over the nodal cubic C with the corkscrew surface, each point of C besides the origin get raised to a height according to the slope of the line between it and the origin. The singularity at the origin gets separated into two points on the corkscrew surface according to its two tangent directions, resolving (desingularizing) the singularity.

We would like to extend this technique to analyze critical points of planar polynomial systems of differential equations. For example, consider the system

$$\begin{aligned} x' &= xy \\ y' &= y^2 - x^4. \end{aligned}$$

The phase portrait is



By inspection, the linearization at the critical point $(0, 0)$ is the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

which clearly doesn't tell us much about the shape of the flow around the critical point.

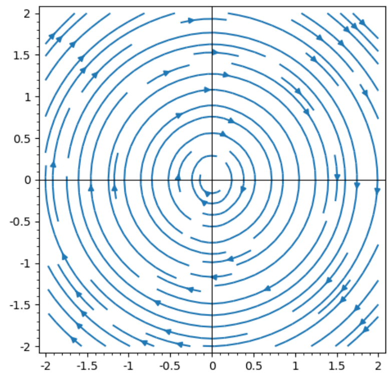
We blow up the critical point using the equation $y = zx$ for the corkscrew surface. We get $y' = z'x + zx'$, which implies

$$\begin{aligned} z' &= \frac{y' - zx'}{x} \\ &= \frac{(y^2 - x^4) - z\,(xy)}{x} \\ &= \frac{(z^2x^2 - x^4) - z^2x^2}{x} \\ &= -x^3. \end{aligned}$$

Notice that by canceling x in the calculation above, we have an equation that is defined at $x = 0$. We've filled in a hole that way. (Recall that in the blow up process, we threw away the z-axis in the intersection of the cylinder with the corkscrew surface). Next,

$$x' = xy = x\,(zx) = x^2z$$

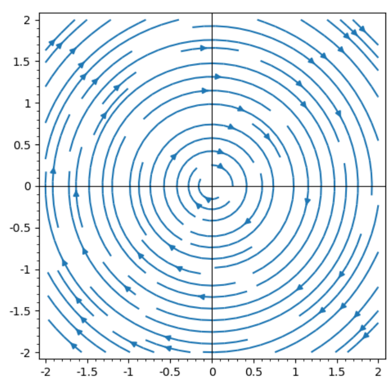
So our system at this point is defined by the vector field $f(x, z) = (x^2z, -x^3)$:



Again, the linearization of this vector field at the origin is given by the zero matrix. However, scaling the vector field by $1/x^2$ at points where $x \neq 0$ gives the vector field $(z, -x)$, which is also defined at $x = 0$: Our final system is

$$
\begin{aligned}
x' &= z \\
z' &= -x,
\end{aligned}
$$

which is a center:



To get back out original vector field, imagine this center wrapped along the corkscrew surface, before projecting to the x, z-plane. Take the flow we would get on the corkscrew surface and project that back down to the x, y-plane: