Math 322 lecture for Monday, Week 12

Today we will do more examples of global phase portraits. First we recall the method.

We are studying a planar system

$$x' = P(x, y) \tag{1}$$
$$y' = Q(x, y) \tag{2}$$

where P and Q are polynomials. Embedded the system in the plane $z = 1$ in $\mathbb{R}^3$ and project the flow along lines centered at the origin onto the unit sphere S centered at the origin. This flow induced a flow along the equator of the sphere. We are interested in critical points of this flow along the equator, where $z = 0$. The correspond to *critical points at infinity* in our original planar system.

Step 1. Let $d := \max\{\deg P, \deg Q\}$. Clear denominators in the equation

$$yP\left(\frac{x}{z}, \frac{y}{z}\right) - xQ\left(\frac{x}{z}, \frac{y}{z}\right) = 0,$$

and then set $z = 0$. Note: we clear denominators by multiplying the above expression through by z^d.

The result is several pairs of antipodal points of the form $(a, b, 0), (-a, -b, 0)$ on the equator of the sphere. The critical points occur among these.

Step 2. Projection to the plane $x = 1$: If $a \neq 0$, to analyze the pair of antipodal points $(a, b, 0)$ and $(-a, -b, 0)$, we may assume $a > 0$. To determine the behavior of our spherical system at $(a, b, 0)$, analyze the point $\left(\frac{b}{a}, 0\right)$ for the system

$$\boxed{\begin{aligned} u' &= v^d\left(Q\left(\frac{1}{v}, \frac{u}{v}\right) - uP\left(\frac{1}{v}, \frac{u}{v}\right)\right) \\ v' &= -v^{d+1}P\left(\frac{1}{v}, \frac{u}{v}\right). \end{aligned}}$$

The behavior at the antipodal point $(-a, -b, 0)$ will be the same except if d is even, the direction of flow is reversed. (See the previous lecture for an explanation.)

Projection to the plane $y = 1$: If $a \neq 0$, then if $(a, b, 0) = (0, b, 0)$ is to be on the sphere, we must have $b = \pm 1$. The behavior at $(0, 1, 0)$ is determined by the behavior of the point $(0, 0)$ in the system

$$\boxed{\begin{aligned} u' &= v^d \left(P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right) \right) \\ v' &= -v^{d+1} Q\left(\frac{u}{v}, \frac{1}{v}\right). \end{aligned}}$$

The behavior at $(0, -1, 0)$ is the same except d is even, the direction of the flow is reversed.

Global phase portrait. Imagine looking down from a point way up on the z-axis at the flow we projected to the sphere. Below us, we see a flow on a disc whose boundary is the equator of the sphere. That's the global phase portrait: the projection of the flow on the upper hemisphere of the sphere down to the $z = 0$ plane. By analyzing the behavior of our planar system at all of its equilibrium points, including those at infinity, we show get a good *qualitative* understanding of the flow of the system from its global phase portrait.

Exercises from last time. Consider the system

$$\begin{aligned} x' &= x^2 + y^2 - 1 \\ y' &= 5xy - 5. \end{aligned}$$

Problem 1. Find all critical points of the system, including critical points at ∞.

Solution. Find the critical points at infinity:

$$\begin{aligned} yP\left(\frac{x}{z}, \frac{y}{z}\right) - xQ\left(\frac{x}{z}, \frac{y}{z}\right) &= y\left(\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 - 1\right) - x\left(5\left(\frac{x}{z}\right)\left(\frac{y}{z}\right) - 5\right) \\ &= \frac{x^2y + y^3 - yz^2 - 5x^2y + 5xz^2}{z^2} \\ &= \frac{-4x^2y + y^3 - yz^2 + 5xz^2}{z^2} \\ &= 0. \end{aligned}$$

Clear denominators:

$$-4x^2y + y^3 - yz^2 + 5xz^2 = 0.$$

Set $z = 0$ to get

$$-4x^2y + y^3 = y(-4x^2 + y^2) = 0.$$

So the points $(x, y, 0)$ at infinity (along the equator of the sphere) occur where $y = 0$ or where $y = \pm 2x$. So we need to consider the three points

$$(1, 0, 0), \quad \frac{1}{\sqrt{5}}(1, \pm 2, 0),$$

and their antipodes. In this case, since $d = \max\{\deg P, \deg Q\} = 2$ is even, the behavior of a pair of antipodes is the same except that the flow is reversed.

Problem 2. Analyze each point at ∞ by projecting to the plane $x = 1$. Draw the flow in the plane $x = 1$.

Solution. We consider the system

$$u' = v^2\left(\frac{5u}{v^2} - 5 - u\left(\frac{1}{v^2} + \frac{u^2}{v^2} - 1\right)\right) = 4u - 5v^2 - u^3 + uv^2$$

$$v' = -v^3\left(\frac{1}{v^2} + \frac{u^2}{v^2} - 1\right) = -v - u^2v + v^3.$$

We are interested in the critical points $(0, 0)$ and $(\pm 2, 0)$ for the system

$$\begin{aligned} u' &= 4u - 5v^2 - u^3 + uv^2 \\ v' &= -v - u^2v + v^3. \end{aligned}$$

The Jacobian matrix for the right-hand side is

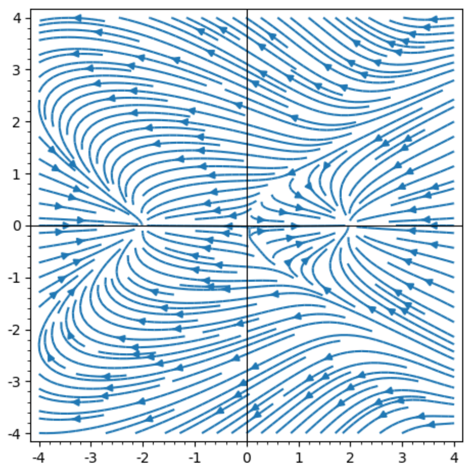
$$Jf := \begin{pmatrix} 4 - 3u^2 + v^2 & -10v + 2uv \\ -2uv & -1 - u^2 + 3v^2 \end{pmatrix}.$$

We have

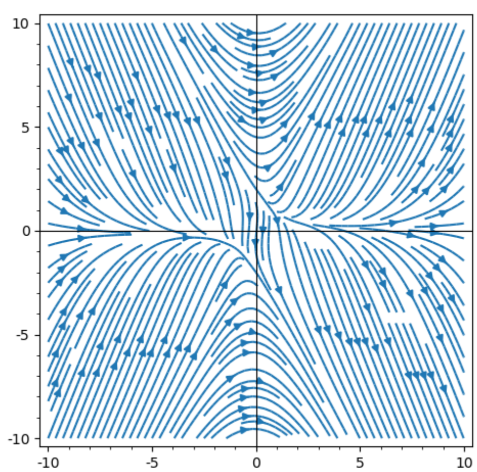
$$Jf(0, 0) = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad J(\pm 2, 0) = \begin{pmatrix} -8 & 0 \\ 0 & -5 \end{pmatrix}.$$

Thus, we get a saddle at $(0, 0)$ and a sinks at $(\pm 2, 1)$.

Here is the flow given by this system in the $x = 1$ plane:



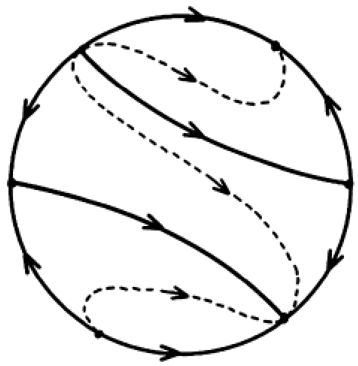
Problem 3. Try to reconcile your results from Problem 2 with the flow of the original system displayed below:

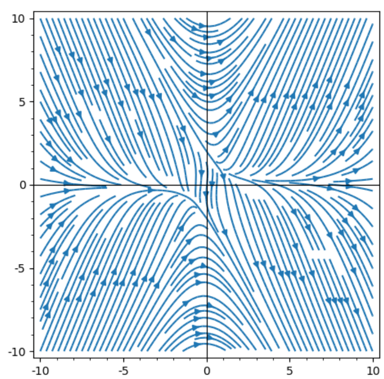


Try to draw a global phase portrait.

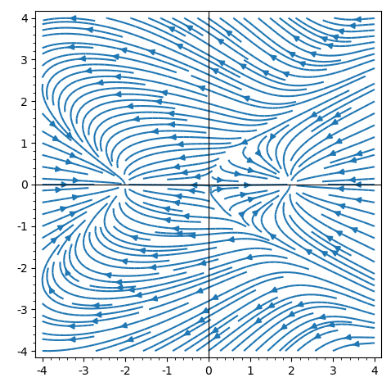
Solution. Notice in the original system, the almost parallel trajectories heading off to the northeast as a slope of about 2. If we consider draping the plane over an underlying sphere, these trajectories will converge to the critical point $(1, 2, 0)$ on the equator. Similarly, the trajectories heading to the southeast with a slope of approximately -2 will converge to $(1, -2, 0)$. The trajectories streaming in on the left from the northwest and southwest are coming from the antipodal points $(-1, 2, 0)$ and $(-1, -2, 0)$.

Global phase portrait:





original system



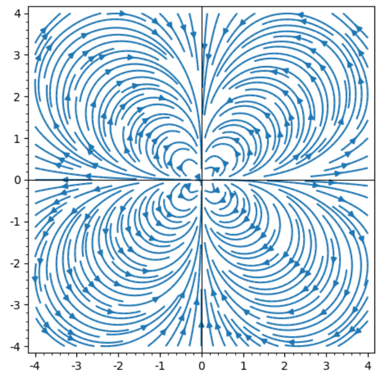
$x = 1$ plane projection

We finish with a couple more examples of global phase portraits.

Example. Consider the system

$$x' = x^3 - 3xy^2$$
$$y' = 3x^2y - y^3.$$

The flow is



To find the interesting points at infinity (along the equator of the sphere) we compute

$$yP\left(\frac{x}{z},\frac{y}{z}\right)-xQ\left(\frac{x}{z},\frac{y}{z}\right)=0,$$

then clear denominators and set $z=0$ to get

$$0=-2x^3y-2xy^3=-2xy(x^2+y^2).$$

There are no points on the equator for which $x^2+y^2=0$, i.e., for which $x=y=0$. So the solutions occur when on of x and y is 0 and the other is nonzero. That gives two pair of antipodal points to check: $\pm(1,0,0)$ and $\pm(0,1,0)$. To check $(1,0,0)$ we project to the $x=1$ plane. The system is

$$\begin{aligned}u'&=2u+2u^3\\ v'&=-v+3u^2v\end{aligned}$$

Linearized at the point $(0,0)$, the system becomes

$$\begin{aligned}u'&=2u\\ v'&=-v,\end{aligned}$$

which is a saddle. Since $d=3$ is odd, the behavior at the antipodal point $(-1,0,0)$ is the same.

For the point $(0,1,0)$ we project to the plane $y=1$. The system is

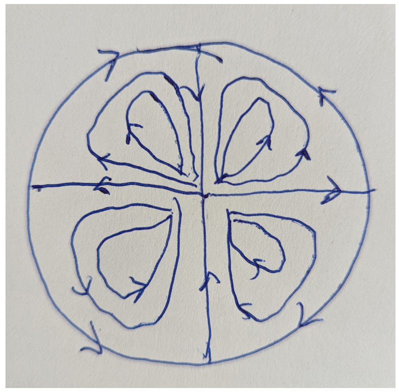
$$\begin{aligned}u'&=-2u-2u^3\\ v'&=v-3u^2v\end{aligned}$$

Linearized at the point $(0,0)$, the system becomes

$$\begin{aligned}u'&=-2u\\ v'&=v,\end{aligned}$$

another saddle. The antipodal point $(0,-1,0)$ is similar.

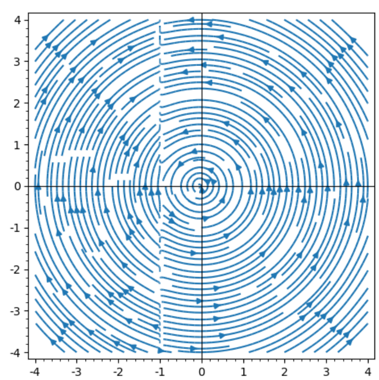
The global phase portrait is:



Example. Consider the system

$$x' = -y - x * y$$
$$y' = x + x^2.$$

The flow is



To find the interesting points at infinity (along the equator of the sphere) we compute

$$yP\left(\frac{x}{z}, \frac{y}{z}\right) - xQ\left(\frac{x}{z}, \frac{y}{z}\right) = 0,$$

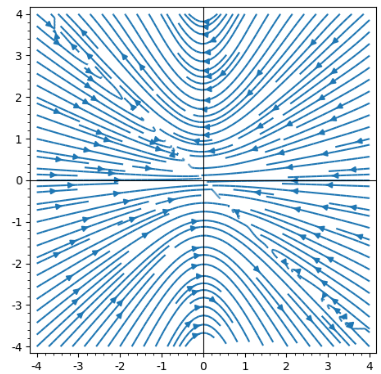
then clear denominators and set $z = 0$ to get

$$0 = -x^3 - xy^2 = -x(x^2 + y^2)$$

So we need $x = 0$. That means we are interested in the points $\pm(0, 1, 0)$ on the equator. Projecting to the plane $y = 1$, we get the system

$$\begin{aligned} u' &= -u - v - u^3 - u^2 v \\ v' &= -u^2 v - uv^2, \end{aligned}$$

which looks like



Note the line of critical points in the planar flow diagram and how those transform when that flow is projected onto the sphere and then onto the plane $x = 1$.