Math 322 lecture for Monday, Week 11

Our goal now is to formally define the index of a vector field on a surface besides $\mathbb{R}^2$. Let S be a 2-dimensional manifold. So $S = \cap_i U_i$ where the U_i open sets and with homeomorphisms

$$h_i \colon U_i \to V_i \subset \mathbb{R}^2$$

which allow us to think of each U_i as an open subset $V_i \subseteq \mathbb{R}^2$ of the plane. On overlaps $U_i \cap U_j$, the *change of coordinates* mapping $h_j \circ h_i^{-1}$ is differentiable. Recall that the pair (U_i, h_i) is called a *chart* and the collection of charts is an *atlas* . Suppose that $S \subseteq \mathbb{R}^n$. A *vector field* on S is a C^1-mapping $f \colon S \to \mathbb{R}^n$ such that $f(p)$ is tangent to S, so there is a curve $\gamma \colon (-1, 1) \to S$ such that $\gamma(0) = p$ and $\gamma'(0) = f(p)$.

To calculate the index of a critical point p of f, i.e., at a point where $f(p) = 0$, we first pick U_i such that $p \in U_i$, and we use h_i to identify f with a vector field on $\mathbb{R}^2$ with critical point $h_i(p)$. In detail, if $q \in U_i$, we pick a curve γ in S passing through q at time 0 and such that $\gamma'(0) = f(q)$. To find the corresponding vector on $\mathbb{R}^2$, we use the composition

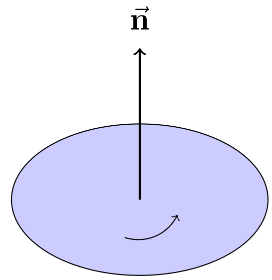
$$(-1, 1) \xrightarrow{\gamma} U_i \xrightarrow{h_i} \mathbb{R}^2.$$

The vector in $\mathbb{R}^2$ at the point q will be $(h_i \circ \gamma)'(0)$.

The next thing we need to define the index of a critical point of a vector field on S is an orientation. An *orientation* is a choice of charts so that the Jacobian of each change of coordinates has positive determinant wherever defined. In other words, for all i, j and for all $p \in U_i \cap U_j$, we assume $\det J(h_j \circ h_i^{-1}) > 0$. Then, to define the *index* of a vector field at a critical point p, first choose a chart (U_i, h_i) with $p \in U_i$. Use h_i to translate the vector field f to a vector field $h_{i,*}(f)$ on $V_i = h_i(U_i)$ as described above. Then define the index $I_f(p)$ to be the index of $h_i(p)$ for the vector field $h_{i,*}(f)$ i.e., $I_f(p) := I_{h_{i,*}(f)}(h_i(p))$.

Not all 2-dimensional manifolds are orientable, for instance, a Möbius strip is not orientable, nor is the projective plane $\mathbb{P}^2$ (which contains a Möbius strip).

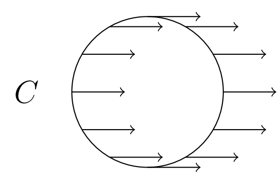
We now assume that S is compact and oriented. It turns out that in this case, we can take S to be a g-hold donut embedded in $\mathbb{R}^3$. We will take the orientation to be the one determined by the outward-pointing normal vector for S: for a chart at a point p, just take a sufficiently small open piece of the surface containing p, and "flatten is out". We take "counterclockwise" to be the direction for which the right-hand rule gives the outward pointing normal vector:



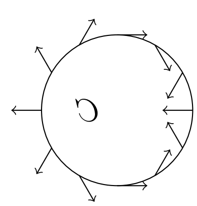
Theorem (Poincaré-Hopf index theorem). Suppose the critical points of the vector field f on S are $p_1, \ldots, p_k$. Then

$$\sum_{i=1}^{k} I_f(S) = 2 - 2g.$$

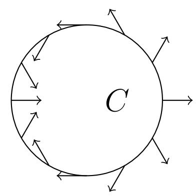
Proof. We first consider the case $g = 0$. So S is an ordinary sphere in $\mathbb{R}^3$. Draw a tiny circle C around a regular point q (i.e., non-critical point). Call the side of C containing q the "inside" of C. We take C tiny enough so that there are no critical points inside C and f is virtually constant around C:



Now for the hard part: imagine cutting out the inside of C, stretching C and the remaining part of S so that this remaining part (containing all of the critical points of f) sits in $\mathbb{R}^2$ with C as its boundary. If you are careful with how the vector field morphs under this transformation, you should get the picture:



The vector field now rotates clockwise twice as we go around C. The critical points are now inside the circle in the picture above, so it looks like the sum of their indices should be 2. However, if you are careful, you'll notice that the orientation has reversed (the normal vectors from the sphere are now pointing into the page). So we should flip the picture, to get the usual orientation:



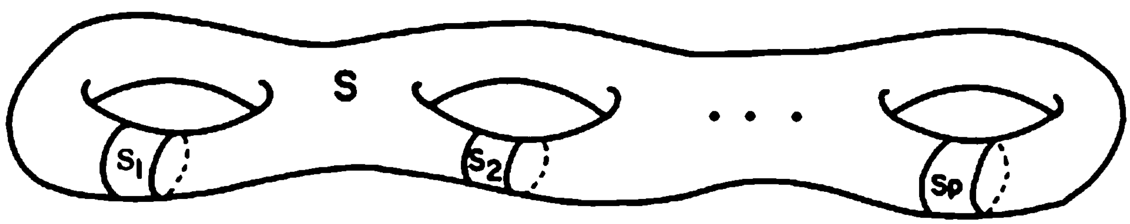
As we travel around the boundary counter-clockwise, the vector field on the boundary rotates twice, again counterclockwise. So the index is still 2 (as opposed to -2).

Thus, the sum of the indices is

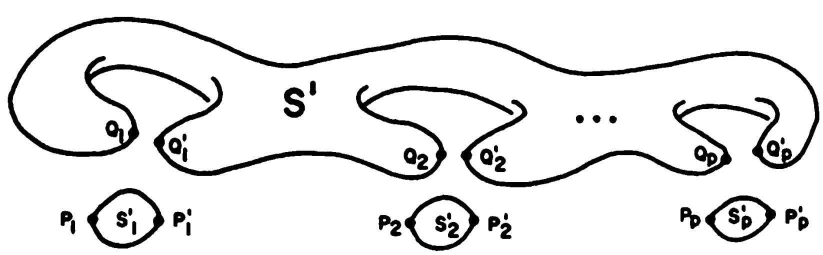
$$2 = 2 - 2 \cdot 0 = 2 - 2g$$

in the case of $S = S^2$.

Now consider a g-holed donut:



The picture is taken from our text, where p is used instead of g. We will keep using g to denote the number of holes. The cylinders $S_1, \ldots, S_g$ are chosen so that they contain no critical points. Next, contract the boundaries of the S_i, morphing the S_i into spheres $S'_1, \ldots, S'_g$:



The vector field morphs, too, creating critical points Q_i and Q'_i for $i = 1, \ldots, g$ on the surface S' pictured above, and corresponding critical points P_i and P'_i on the spheres S_i'. Since the vector field at each P_i is the negative of that on Q_i and similarly for P_i' and Q_i', we have

$$I_f(P_i) = I_f(Q_i) \quad \text{and} \quad I_f(P_i') = I_f(Q_i').$$

Since the S_i' are spheres, from our previous work it follows that

$$I_f(P_i) + I_f(P_i') = 2$$

for all i. The surface S' is a sphere. Therefore,

$$I_f(S') = 2.$$

The result follows:

$$\begin{aligned} I_f(S) &= I_f(S') - \sum_{i=1}^{g} (I_f(Q_i) + I_f(Q_i')) \\ &= I_f(S') - \sum_{i=1}^{p} (I_f(P_i) + I_f(P_i')) \\ &= 2 - 2g. \end{aligned}$$

$\square$

Note: The Poincaré-Hopf theorem also holds for non-orientable manifolds. See our text for a proof (that builds on the above proof).