Math 322 lecture for Friday, Week 11

Recall the notation from last time. We are studying a planar system

$$x' = P(x, y) \tag{1}$$
$$y' = Q(x, y) \tag{2}$$

where P and Q are polynomials. We embedded our system in the plane $z = 1$ in $\mathbb{R}^3$ and projected the flow along lines centered at the origin onto the unit sphere S centered at the origin. This flow induced a flow along the equator of the sphere. We are interested in critical points of this flow along the equator, where $z = 0$. To find those we saw that we should . . .

Step 1. Clear denominators in the equation

$$yP\left(\frac{x}{z}, \frac{y}{z}\right) - xQ\left(\frac{x}{z}, \frac{y}{z}\right) = 0,$$

and then set $z = 0$.

The result is several points of the form $(a, b, 0)$ on the equator of the sphere.

Step 2. To analyze these, we will project the flow on the sphere to either the plane $x = 1$ or the plane $y = 1$. If a, b are both nonzero, then either plane will do. If $a = 0$, then the point in question is $(0, 1, 0)$, and we'd need to project to the plane $y = 1$, and if $b = 0$, the point is $(1, 0, 0)$, and we'd need to project to $x = 1$. We'll consider these cases separately:

Projection to the plane $x = 1$. We project the point $(x, y, 1)$ along a line through the origin, i.e., we scale this point, to get a point on the plane $x = 1$:

$$(x, y, 1) \rightsquigarrow \left(1, \frac{y}{x}, \frac{1}{x}\right).$$

Let

$$u := \frac{y}{x} \quad \text{and} \quad v := \frac{1}{x}.$$

We use the coordinates to identify the plane $x = 1$ with the ordinary plane $\mathbb{R}^2$. It follows that

$$x = \frac{1}{v} \quad \text{and} \quad y = \frac{u}{v}.$$

Plug these into system 1:

$$\begin{aligned} x' = P(x,y) \quad &\Rightarrow \quad \left(\frac{1}{v}\right)' = P\left(\frac{1}{v}, \frac{u}{v}\right) \\ &\Rightarrow \quad v' = -v^2 P\left(\frac{1}{v}, \frac{u}{v}\right) \end{aligned}$$

Similarly,

$$\begin{aligned} y' = Q(x,y) \quad &\Rightarrow \quad \left(\frac{u}{v}\right)' = Q\left(\frac{1}{v}, \frac{u}{v}\right) \\ &\Rightarrow \quad u'v - uv' = v^2 Q\left(\frac{1}{v}, \frac{u}{v}\right) \\ &\Rightarrow \quad u'v = v^2 Q\left(\frac{1}{v}, \frac{u}{v}\right) + uv' \\ &\Rightarrow \quad u'v = v^2 Q\left(\frac{1}{v}, \frac{u}{v}\right) - uv^2 P\left(\frac{1}{v}, \frac{u}{v}\right) \\ &\Rightarrow \quad u' = v\left(Q\left(\frac{1}{v}, \frac{u}{v}\right) - uP\left(\frac{1}{v}, \frac{u}{v}\right)\right). \end{aligned}$$

So the system in the u, v-plane is

$$\begin{aligned} u' &= v\left(Q\left(\frac{1}{v}, \frac{u}{v}\right) - uP\left(\frac{1}{v}, \frac{u}{v}\right)\right) \\ v' &= -v^2 P\left(\frac{1}{v}, \frac{u}{v}\right). \end{aligned} \tag{3}$$

The problem is that it is likely this system is not defined where $v = 0$ (at the equator). To get the induced flow on the equator, we need to clear denominators (thus, scaling the vector field but not changing its direction at any point). Define

$$d := \max\{\deg P, \deg Q\}$$

To clear denominators we scale the vector field in (3) by v^{d-1} to get the system

$$\boxed{\begin{aligned} u' &= v^d\left(Q\left(\frac{1}{v}, \frac{u}{v}\right) - uP\left(\frac{1}{v}, \frac{u}{v}\right)\right) \\ v' &= -v^{d+1} P\left(\frac{1}{v}, \frac{u}{v}\right). \end{aligned}}$$

We analyze the point $\left(\frac{b}{a}, 0\right)$, since this is the point corresponding to $(a, b, 0)$ in the u, v-plane.

Important points: The right side of the system (3) defines the vector field whose trajectories we would like to determine. What effect does scaling that vector field by v^{d-1} have on the solution trajectories? The vector field gives the tangent vector for a solution trajectory. So one effect is to scale the speed of the trajectory by the magnitude $|v|^{d-1}$. Note that $v = 1/x$ where x comes from the point $(x, y, 1)$ in the $z = 1$ plane. As we go "out to infinity" in the $z = 1$ plane, by taking x larger, the scaling factor $|v^{d-1}|$ decreases in magnitude. We have chosen d just write so that the resulting vector field does not blow up on the equator and is also not identically the zero on the equator.

What about the direction of the trajectory? Since both components of the vector field are scaled the same amount, there are two choices: (i) if $v^{d-1} > 0$, the direction is the same, and (ii) if $v^{d-1} < 0$, the direction is reversed. Next, what significance does this have for analyzing critical points at the equator? Suppose we are interesting in a trajectory corresponding containing a point $(x, y, 1)$ in the original $z = 1$ plane. If $x > 0$, then since $v = 1/x > 0$, it follows that $v^{d-1} > 0$, and the direction does not change. On the other hand, if $x < 0$, then $v = 1/x < 0$. If d is odd, then $v^{d-1} > 0$, and if d is even, then $v^{d-1} < 0$. So in the latter case, in which d is even, the direction of the vector field and hence the direction of it solution trajectories is reversed.

Projection to the plane $y = 1$. By a similar analysis (which will be assigned for homework), if $b \neq 0$, we can project to the $y = 1$ plane and derive an analogous system of equations:

$$\boxed{\begin{aligned} u' &= v^d \left(P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right) \right) \\ v' &= -v^{d+1} Q\left(\frac{u}{v}, \frac{1}{v}\right). \end{aligned}}$$

We are interested in the point $\left(\frac{a}{b}, 0\right)$ in this plane.

Global phase portrait A couple of lectures ago, we introduced the global phase portrait of a planar system. It is the central projection of the flow of the vector field onto the top half of the sphere. In order to compute it, find and analyze all critical points of the planar system. Next, find all critical points of the system at infinity. These come in antipodal pairs: $(a, b, 0)$ and $(-a, -b, 0)$. Without loss of generality, suppose $a > 0$. Then we analyze the scaled system, scaling by v^{d-1} with $v = 1/x$

in the plane $x = 1$ at the point $\left(\frac{b}{a}, 0\right)$. At the antipodal point, we analyze the same system but scaled by $(-1/x)^{d-1}$ and at the point $\left(\frac{-b}{-a}, 0\right)$, i.e., at the same point. This means the systems at antipodal points are either the same when projected to the $x = 1$ plane up to a possible reversal of directions (which happens exactly in the case d is even.

Exercises

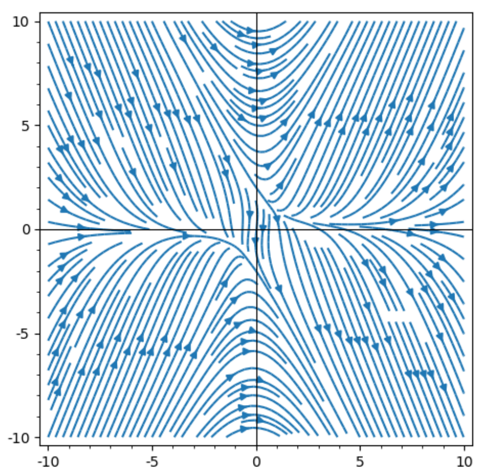
In the following exercises, we will analyze the critical points, both finite and at ∞ for the system

$$
\begin{aligned}
x' &= x^2 + y^2 - 1 \\
y' &= 5xy - 5.
\end{aligned}
\tag{4}
$$

Problem 1. Find all critical points of the system, including critical points at ∞.

Problem 2. Analyze each point at ∞ by projecting to the plane $x = 1$. Draw the flow in the plane $x = 1$.

Problem 3. Try to reconcile your results from Problem 2 with the flow of the original system displayed below:



Try to draw a global phase portrait.