Math 322 lecture for Wednesday, Week 10

GLOBAL THEORY FOR NONLINEAR SYSTEMS: INDEX THEORY

A *Jordan curve* C is the injective continuous image $\gamma\colon S^1 \to \mathbb{R}^2$ of a circle into the plane. Equivalently, it is the continuous image of an interval $\gamma\colon [0,1] \to \mathbb{R}^2$ that is injective on $[0,1)$ and such that $\gamma(0) = \gamma(1)$. The Jordan curve theorem (first conjectured by Bolzano) states that a Jordon curve divides the plane into two connected components. We will impose the further condition that γ be piecewise smooth (continuous derivatives except at a finite number of points).

Let $f(x,y) = (P(x,y), Q(x,y))$ be a smooth vector field in the plane, and let C be a Jordan curve. A *critical point* for f is a point (x_0, y_0) where $f(x_0,y_0) = 0$. (Thus, a critical point would be an equilibrium point for the corresponding system of differential equations.)

Definition. The index $I_f(C)$ of C relative to f is

$$I_f(C) := \frac{\Delta\theta}{2\pi}$$

where $\Delta\theta$ is the change in angle of $f(x,y)$ as (x,y) travels around C counterclockwise.

Exercises.

1. For each of the following vector fields, (i) draw the flow near the origin and draw a circle C containing the origin; (ii) pick some points on C, and draw each point as a vector (with tail at the origin) on a separate picture of $\mathbb{R}^2$; (iii) compute the index:

   (a) $f(x,y) = (-1,-1)$ (b) $f(x,y) = (-x,-y)$

   (c) $f(x,y) = (-y,x)$ (d) $f(x,y) = (-x,y)$.

2. How does the index change in (a)–(d) if f is replaced by $-f$?

3. How would the index change if C were replaced by an ellipse?

Calculation of the index. Let $\gamma(t) = (x(t), y(t))$ be a parametrization of C. By translating, if necessary, we may assume the origin is in the interior of C. Consider the composition of mappings

$$[0,1) \xrightarrow{\gamma} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}^2.$$

We are interested in the change in the angle of $f \circ \gamma$ as t goes from 0 to 1. Write $f \circ \gamma$ in polar coordinates:

$$P(x,y) = r\cos(\theta) \qquad Q(x,y) = r\sin(\theta)$$

where x, y, r, θ are functions of t. Then

$$P' = r'\cos(\theta) - r\theta'\sin(\theta)$$
$$Q' = r'\sin(\theta) + r\theta'\cos(\theta),$$

and it's easy to check that

$$r^2\theta' = PQ' - QP'$$

where $r^2 = P^2 + Q^2$. Therefore, the change in angle is

$$\Delta\theta = \int_{t=0}^{1} \frac{PQ' - QP'}{P^2 + Q^2}\, dt = \int_{t=0}^{1} (P,Q) \cdot \left(\frac{Q'}{P^2+Q^2}, -\frac{P'}{P^2+Q^2} \right) dt = \oint_C \frac{PdQ - QdP}{P^2 + Q^2}.$$

So the index is

$$I_f(C) = \frac{\Delta\theta}{2\pi} = \frac{1}{2\pi} \oint_C \frac{PdQ - QdP}{P^2 + Q^2}.$$

To convert to the language of differential forms, let

$$\omega := \frac{x\,dy - y\,dx}{x^2 + y^2}$$

be the "flow form" for the circular vector field $\left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$ on $\mathbb{R}^2$. Then the index is calculated by integrating the pullback of ω along f over C:

$$I_f(C) = \frac{1}{2\pi} \oint_\gamma f^*\omega.$$

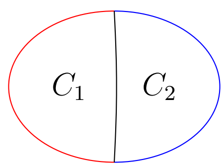
Example. Let $f(x,y) = (-y, x)$, and let C be the unit circle centered at the origin parametrized by $\gamma(t) = (\cos(t), \sin(t))$ for $t \in [0, 2\pi]$. Then

$$f(\gamma(t)) = (-\sin(t), \cos(t)).$$

$$
\begin{aligned}
I_f(C) &= \frac{\Delta\theta}{2\pi} = \frac{1}{2\pi}\oint_C \frac{PdQ - QdP}{P^2+Q^2} \\
&= \frac{1}{2\pi}\int_C (P,Q)\cdot\left(\frac{Q'}{P^2+Q^2}, -\frac{P'}{P^2+Q^2}\right) dt \\
&= \frac{1}{2\pi}\int_{t=0}^{1} (-\sin(t),\cos(t))\cdot\left(\frac{-\sin(t)}{(-\sin(t))^2+\cos(t)^2}, -\frac{-\cos(t)}{(-\sin(t))^2+\cos(t)^2}\right) dt \\
&= \frac{1}{2\pi}\int_{t=0}^{2\pi} dt = 1.
\end{aligned}
$$

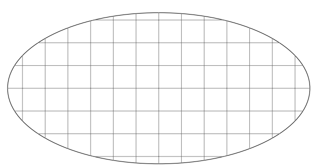
Theorem. If there are no critical points on C or in its interior, then $I_f(C) = 0$.

Proof. **Step 1.** Suppose $C = C_1 + C_2$ as shown below:

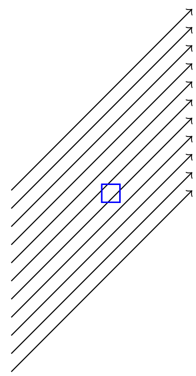


The curves C_1 and C_2 share the vertical middle line. In the calculating of the sum of the indices of f relative to C_1 and to C_2, the contribution from the middle line cancels (imaging traveling along both C_1 and C_2 in the counterclockwise direction). Thus, $I_f(C) = I_f(C_1) + I_f(C_2)$.

Step 2. Next, divide C into a sum of lots of tiny closed curves:



So $C = C_1 + \cdots + C_n$. It suffices to show that $I_f(C_i) = 0$ for all i. Some details and a reference are provided below, but the main idea is that since $f \neq \vec{0}$ on or inside C, by taking C_i small enough, the vector field's angle cannot change much along C_i:



Let X denote C union its interior. Since X is compact, and the component functions P and Q of the vector field are continuous, it follows that P and Q are uniformly continuous. That means that given any $\varepsilon > 0$, we can make the widths of the C_i simultaneously small enough to that P and Q change by a value less then ε on each C_i. Also, since X is compact and f is continuous and nonzero in X, the value of $|f(x, y)|$ attains a nonzero minimum on X. This means that it is possible to take the widths of the C_i simultaneously small enough so that the angle of f on X varies by only a small amount (less than 2π is sufficient). Some details appear in our text, Problem 2, Chapter 3.

The result then follows from Step 1: $I_f(C) = \sum_i I_f(C_i) = \sum_i 0 = 0$. □

Corollary. Let C be a Jordan curve. Suppose there are no critical points on C but that there may be critical points in its interior. Let C' a Jordan curve in the interior of C, and suppose there are no critical points on C', and there are no critical points in the region between C and C'. Then $I_f(C) = I_f(C')$.

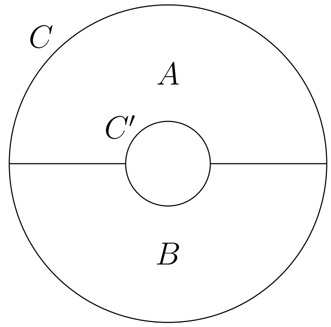
Proof. Referring to the diagram below, let ∂A and ∂B be the Jordan curves forming the boundaries of the closed regions labeled A and B. So both ∂A and ∂B are the boundaries of deformed rectangles. Imagine traveling counterclockwise along these curve, computing their indices. You should see that

$$I_f(\partial A + \partial B) = I_f(C) - I_f(C').$$

However, since there are no critical points in A and none in B, using the previous theorem, we have

$$I_f(\partial A + \partial B) = I_f(\partial A) + I_f(\partial B) = 0.$$

The result follows.



□

Corollary. If C and C' are Jordan curves containing the same finite set of critical points in their interiors, then $I_f(C) = I_f(C')$.

Proof. Let D be a Jordan curve containing all the critical points and contained in the interiors of both C and C'. Then by the previous corollary,

$$I_f(C) = I_f(D) = I_f(C').$$

□

Definition. Let p be an isolated critical point of f. Define the *index of x relative to f* to be

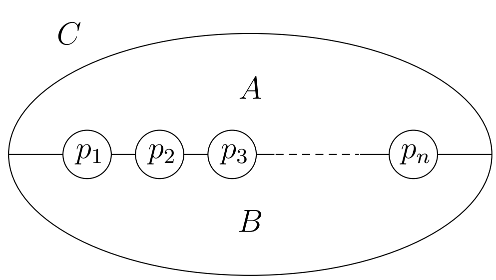
$$I_f(p) := I_f(C)$$

where C is any Jordan curve containing p as its only interior critical point. (This is well-defined from the previous corollary.)

Theorem. Let $p_1, \dots, p_n$ be the critical points inside C. Then

$$I_f(C) = \sum_{i=1}^{n} I_f(p_i).$$

Proof. The proof is similar to that of our first corollary:



We have

$$0 = I_f(\partial A + \partial B) = I_f(C) - \sum_{i=1}^{n} I_f(p_i).$$

$\square$