Math 322 lecture for Friday, Week 9

LIAPUNOV FUNCTIONS

Theorem. Let $f \in C^1(E)$ and $f(x_0) = 0$. Let $V\colon E \to \mathbb{R}$ also be C^1 (continuously differentiable). Suppose that $V(p) \geq 0$ and $V(p) = 0$ if and only if $p = x_0$. Then:

1. If $\dot{V}$ is negative semidefinite ($\dot{V}(p) \leq 0$ for all $p \in E \setminus \{x_0\}$) then x_0 is stable.
2. If $\dot{V}$ is negative definite ($\dot{V}(p) < 0$ for all $p \in E \setminus \{x_0\}$) then x_0 is asymptotically stable.
3. If $\dot{V}$ is positive definite ($\dot{V}(p) > 0$ for all $p \in E \setminus \{x_0\}$), then x_0 is unstable.

Proof. As before, we may assume $x_0 = (0, 0)$ is the equilibrium point. Part (1) was proved in the last lecture.

(2) Last time, we were in the midst of proving part (2). Using the notation from last time, so far, we have shown that for every sequence $t_1 < t_2 < \cdots$ such that $\lim_{n\to\infty} t_n = \infty$, there exists a subsequence $\{t_{n_k}\}$ such that $\lim_{k\to\infty} \phi(t_{n_k}, p) = 0 \in \mathbb{R}^n$. We now need to show $\lim_{t\to\infty} \phi(t, p) = x_0 = 0$. If not, there exists an $\eta > 0$ such that for all n, there exists $t_n > n$ such that

$$|\phi(t_n, p)| \geq \eta > 0. \tag{1}$$

We may assume that the sequence t_n is increasing. However, by Bolzano-Weierstrass, there again exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that $\phi(t_{n_k}, p)$ converges, and as we have seen, it must converge to 0. But that's impossible in light of (1).

(3) Finally, now suppose that $\dot{V}(p) > 0$ for all $p \in E \setminus \{x_0\}$. Choose $\varepsilon > 0$ such that $\overline{B_\varepsilon(0)} \subset E$. We'll show that given any point $p \in B_\varepsilon(0) \setminus \{0\}$, we have that $\phi_t(p)$ leaves $B_\varepsilon(0)$ at some point, i.e., there exists $t \geq 0$ such that $|\phi_t(p)| \geq \varepsilon$.

Given $p \in B_\varepsilon(0) \setminus \{0\}$, since V is strictly increasing on trajectories,

$$V(\phi_t(p)) > V(\phi_0(p)) = V(p) > 0$$

for all $t > 0$. Thus, $\phi_t(p)$ is bounded away from 0. Say $|\phi_t(p)| \geq \eta > 0$ for all $t \geq 0$. Since $\eta \leq |\phi_0(p)| = |p| < \varepsilon$, it follows that $\eta < \varepsilon$. Define

$$m := \min_{y:\eta \leq |y| \leq \varepsilon} \dot{V}(y),$$

which exists since $\dot{V}$ is continuous and y is restricted to a compact set. In fact, for that same reason, $m = V(q)$ for some point in the set over which we are minimized.

Therefore, $m > 0$. Supposing for contradiction that $\phi_t(p)$ stays inside $B_\varepsilon(x_0)$ for all $t \geq 0$, we have $\dot{V}(\phi_t(p) \geq m$ for all $t \geq 0$. Hence,

$$V(\phi_t(p)) - V(p) = V(\phi_t(p)) - V(\phi_0(p)) = \int_{s=0}^{t} \dot{V}(\phi_s(p))\, ds \geq mt \to \infty$$

as $t \to \infty$. But since V is continuous, it achieves a maximum on $\overline{B_\varepsilon(x_0)}$—a contradiction. $\square$

Example. Consider the system

$$\begin{aligned} x' &= -2y + yz \\ y' &= x - xz \\ z' &= xy. \end{aligned}$$

and the Jacobian at the origin is

$$J(0) = \begin{pmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The characteristic polynomial is

$$\det \begin{pmatrix} -x & -2 & 0 \\ 1 & -x & 0 \\ 0 & 0 & -x \end{pmatrix} = -x^3 - 2x = -x(x^2 + 2).$$

So the eigenvalues are $0, \pm\sqrt{2}i$. So the origin is a nonhyperbolic equilibrium point. To determine stability, we look for a suitable Liapunov function. We guess a function of the form

$$V = ax^2 + by^2 + cz^2$$

with positive constants a, b, c. We have

$$\begin{aligned} \dot{V} &= 2axx' + 2byy' + 2czz' \\ &= 2ax(-2y + yz) + 2by(x - xz) + 2cz(xy) \\ &= 2(-2a + b)xy + (a - b + c)xyz. \end{aligned}$$

Take $a = c = 1$ and $b = 2$, and we get $V = x^2 + 2y^2 + z^2$ with $\dot{V} = 0$. This means that trajectories stay on the ellipsoids that are level sets of V.