Math 322 worksheet for Wednesday, Week 8

Recall the algorithm for creating the stable and unstable manifolds for an n-dimensional system $x' = f(x)$ as an equilibrium point x_0 at which Df_{x_0} has k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part:

(a) Preprocessing:

(a) If $x_0 \neq 0$, substitute $x - x_0$ for x. Then define $F(x) = f(x) - Jf(0)x$ to write the system as

$$x' = Jf(0)x + F(x).$$

(b) Choose $P \in M_n(\mathbb{R})$ such that

$$P^{-1}Jf(0)P = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where A has k eigenvalues with negative real part and B has $n-k$ eigenvalues with positive real part. Define

$$U(t) := \begin{pmatrix} e^{At} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad V(t) := \begin{pmatrix} 0 & 0 \\ 0 & e^{Bt} \end{pmatrix}$$

so that

$$\exp\left(\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) t\right) = U(t) + V(t).$$

(c) Let $y = P^{-1}x$ and $G(y) := P^{-1}F(Py)$ to get the system

$$y' = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} y + G(y).$$

(b) Apply the method of successive approximations by iterating

$$(Tu)(t,a) := U(t)a + \int_{s=0}^{t} U(t-s)G(u(s,a))\,ds - \int_{s=t}^{\infty} V(t-s)G(u(s,a))\,ds.$$

where $a \in \mathbb{R}^n$ starting at $u^{(0)}(t,a) = 0$.

(c) A stable manifold is given as the set of points

$$(a_1, \ldots, a_k, u_{k+1}(0, a_1, \ldots, a_k, 0, \ldots, 0), \ldots, u_n(0, a_1, \ldots, a_k, 0, \ldots, 0))$$

as $(a_1, \ldots, a_k)$ varies in a neighborhood of the origin in $\mathbb{R}^k$.

(d) For the unstable manifold, replace t by $-t$ in the original system to get

$$y' = -\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} y - G(y)$$

Make the swap $\phi\colon y \mapsto (y_{k+1}, \ldots, y_n, y_1, \ldots, y_k)$ to get the system

$$(\phi(y))' = \begin{pmatrix} -B & 0 \\ 0 & -A \end{pmatrix} \phi(y) - \phi(G(y)),$$

apply the method of successive approximations, then swap back, applying ϕ^{-1} to the points in the resulting manifold.

Problem 1. Apply the algorithm just outlined to the system

$$\begin{aligned} x' &= -x - y^2 \\ y' &= y + x^2 \end{aligned}$$

to approximate stable and unstable manifolds at the equilibrium point $x_0 = (0,0)$.

Preprocessing.

$$Jf(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad F(x,y) = f(x,y) - Jf(0,0)(x,y) = (-y^2, x^2)$$

$$P = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \end{pmatrix}$$

$$U(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & 0 \end{pmatrix}, \qquad V(t) = \begin{pmatrix} 0 & 0 \\ 0 & e^{t} \end{pmatrix}$$

$$G(x,y) = F(x,y) = (-y^2, x^2)$$

The transformed system is:

$$y' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -y^2 \\ x^2 \end{pmatrix}$$

Successive approximations. To approximate a stable manifold, iterate

$$(Tu)(t,a) = U(t)a + \int_{s=0}^{t} U(t-s)G(u(s,a))\,ds - \int_{s=t}^{\infty} V(t-s)G(u(s,a))\,ds$$

where $a=(a_1,a_2)$ starting at $u^{(0)}(t,a)=(0,0)$.

$$\begin{aligned} u^{(1)}(t,a) &= \begin{pmatrix} e^{-t} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \int_{s=0}^{t} \begin{pmatrix} e^{s-t} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} ds - \int_{s=t}^{\infty} \begin{pmatrix} 0 & 0 \\ 0 & e^{t-s} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} ds \\ &= \begin{pmatrix} a_1 e^{-t} \\ 0 \end{pmatrix}. \end{aligned}$$

$$\begin{aligned} u^{(2)}(t,a) &= \begin{pmatrix} a_1 e^{-t} \\ 0 \end{pmatrix} + \int_{s=0}^{t} \begin{pmatrix} e^{s-t} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ a_1^2 e^{-2s} \end{pmatrix} ds - \int_{s=t}^{\infty} \begin{pmatrix} 0 & 0 \\ 0 & e^{t-s} \end{pmatrix}\begin{pmatrix} 0 \\ a_1^2 e^{-2s} \end{pmatrix} ds \\ &= \begin{pmatrix} a_1 e^{-t} \\ 0 \end{pmatrix} - \int_{s=t}^{\infty} \begin{pmatrix} 0 \\ a_1^2 e^{t-3s} \end{pmatrix} ds \\ &= \begin{pmatrix} a_1 e^{-t} \\ 0 \end{pmatrix} - \left(\begin{pmatrix} c \\ -\frac{1}{3}a_1^2 e^{t-3s} \end{pmatrix}\Bigg|_{s=t}^{\infty} \right) \\ &= \begin{pmatrix} a_1 e^{-t} \\ -\frac{1}{3}a_1^2 e^{-2t} \end{pmatrix}. \end{aligned}$$

$$\begin{aligned} u^{(3)}(t,a) &= \begin{pmatrix} a_1 e^{-t} \\ 0 \end{pmatrix} + \int_{s=0}^{t} \begin{pmatrix} e^{s-t} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -\left(-\frac{1}{3}a_1^2 e^{-2s}\right)^2 \\ (a_1 e^{-s})^2 \end{pmatrix} ds \\ &\qquad - \int_{s=t}^{\infty} \begin{pmatrix} 0 & 0 \\ 0 & e^{t-s} \end{pmatrix}\begin{pmatrix} -\left(-\frac{1}{3}a_1^2 e^{-2s}\right)^2 \\ (a_1 e^{-s})^2 \end{pmatrix} ds \\ &= \begin{pmatrix} a_1 e^{-t} \\ 0 \end{pmatrix} + \int_{s=0}^{t} \begin{pmatrix} e^{s-t} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -\frac{1}{9}a_1^4 e^{-4s} \\ a_1^2 e^{-2s} \end{pmatrix} ds \\ &\qquad - \int_{s=t}^{\infty} \begin{pmatrix} 0 & 0 \\ 0 & e^{t-s} \end{pmatrix}\begin{pmatrix} -\frac{1}{9}a_1^4 e^{-4s} \\ a_1^2 e^{-2s} \end{pmatrix} ds \\ &= \begin{pmatrix} a_1 e^{-t} \\ 0 \end{pmatrix} + \int_{s=0}^{t} \begin{pmatrix} -\frac{1}{9}a_1^4 e^{-t-3s} \\ 0 \end{pmatrix} ds - \int_{s=t}^{\infty} \begin{pmatrix} 0 \\ a_1^2 e^{t-3s} \end{pmatrix} ds \end{aligned}$$

$$= \begin{pmatrix} a_1 e^{-t} \\ 0 \end{pmatrix} + \left(\left. \begin{pmatrix} \frac{1}{27} a_1^4 e^{-t-3s} \\ 0 \end{pmatrix} \right|_{s=0}^{t} \right) - \left(\left. \begin{pmatrix} c \\ -\frac{1}{3} a_1^2 e^{t-3s} \end{pmatrix} \right|_{s=t}^{\infty} \right)$$

$$= \begin{pmatrix} a_1 e^{-t} \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{27} a_1^4 e^{-4t} \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{27} a_1^4 e^{-t} \\ 0 \end{pmatrix} - \left(- \begin{pmatrix} 0 \\ -\frac{1}{3} a_1^2 e^{-2t} \end{pmatrix} \right)$$

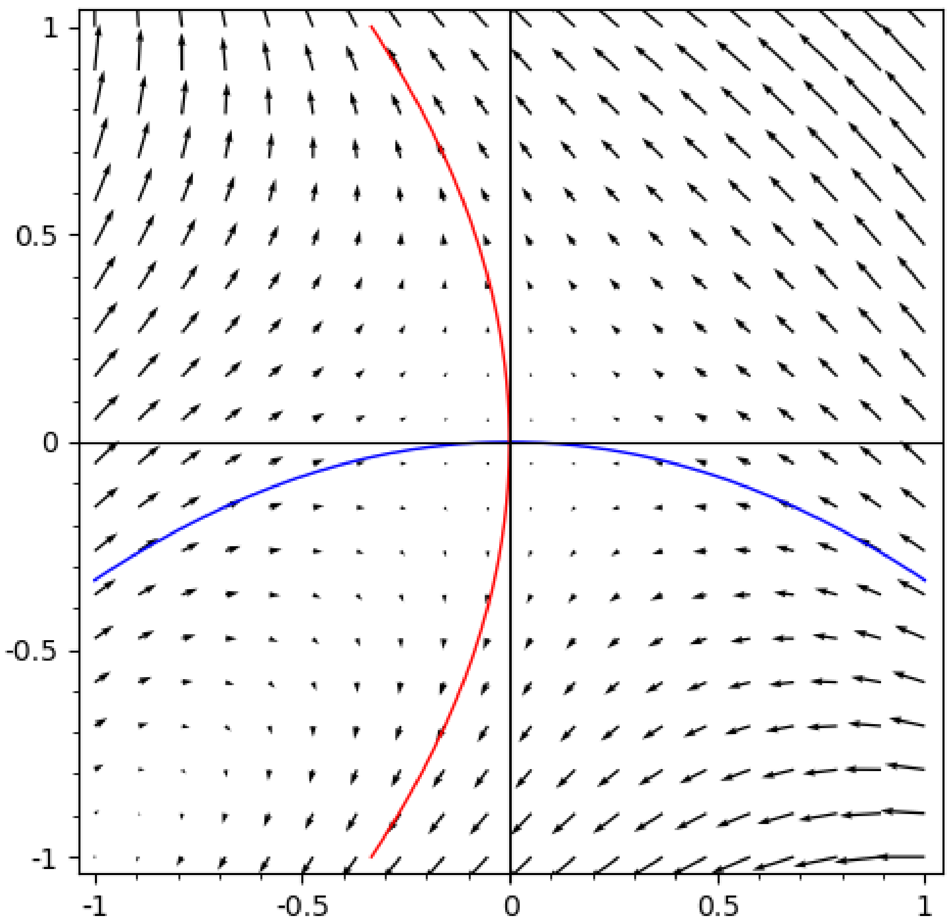
$$= \begin{pmatrix} a_1 e^{-t} + \frac{1}{27} a_1^4 e^{-4t} - \frac{1}{27} a_1^4 e^{-t} \\ -\frac{1}{3} a_1^2 e^{-2t} \end{pmatrix}.$$

An approximation of the stable manifold is given by:

$$(a_1, u_2^{(3)}(0, a_1, 0)) = \left(a_1, -\frac{1}{3} a_1^2 \right)$$

Draw your approximation of the stable manifold on the vector field:

Stable in blue and unstable in red:



Two problems to consider if you have extra time:

I. Approximate the unstable manifold: replace t by $-t$, make the swap $\phi\colon (x,y) \mapsto (y,x)$, and consider the system system

$$\begin{pmatrix} y' \\ x' \end{pmatrix} = \begin{pmatrix} -B & 0 \\ 0 & -A \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} - \phi(G(x,y)).$$

After the swap, you should get the same system you considered for the stable manifold but with x and y swapped. Add the approximation of the unstable space to your drawing.

The unstable manifold is approximated by

$$a_1 = -\frac{a_2^2}{3}.$$

II. Find the other equilibrium point for our system. What can you say about stable and unstable manifolds near this point? What is the behavior of the flow near this point?

To find the equilibrium points, we need to solve the system

$$\begin{aligned} -x - y^2 &= 0 \\ y + x^2 &= 0. \end{aligned}$$

The first equation says $x = -y^2$. Substituting this into the second equation gives $y + x^2 = y + y^4 = 0$, which implies $y = 0$ or $y^3 = -1$. So the equilibrium points are $(0, 0)$ and $(-1, -1)$. Since

$$Jf(x, y) = \begin{pmatrix} -1 & -2y \\ 2x & 1 \end{pmatrix} \quad \Rightarrow \quad Jf(-1, -1) = \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}.$$

The determinant is $\delta = 3$ and the trace is $\tau = 0$. So the discriminant is $\sqrt{\tau^2 - 4\delta} = \pm\sqrt{3}$. So the linearized system is a center. The stable manifold theorem does not apply since its condition on the eigenvalues is not met. It's hard to say what happens for the nonlinear system: