Math 322 worksheet for Wednesday, Week 7

Definition. An *equilibrium point* for a system of differential equations in $\mathbb{R}^n$

$$x' = f(x)$$

is a point $p \in \mathbb{R}^n$ such that $f(p) = 0$.

The reason for the terminology is that if p is an equilibrium point then a solution (*the solution* if f is continuously differentiable) with initial condition $x(0) = p$ is the constant solution $x(t) = p$.

We hope to get a qualitative sense of the solutions to our system near an equilibrium point p by replacing the system with a linear approximation:

$$x' = Jf_p$$

where Jf_p is the Jacobian matrix for f at p.

Consider the system of equations

$$\begin{aligned} x' &= (x^2 - 1)y \\ y' &= (1 - y^2)\left(x + \frac{3}{10}y\right). \end{aligned}$$

So in this case $f(x, y) = ((x^2 - 1)y, (1 - y^2)\left(x + \frac{3}{10}y\right))$.

Problem 1. Find all equilibrium points for the system and plot them in the plane.

Problem 2. Compute the Jacobian matrix $Jf_{(x,y)}$ for our f at an arbitrary point (x, y).

Problem 3 For each equilibrium point p, analyze the linear system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = Jf_p \begin{pmatrix} x \\ y \end{pmatrix}$$

by looking at the eigenvalues of Jf_p. Do you get a saddle? A stable focus or node? An unstable focus or node? A center? (See the last page for a quick guide.)

Problem 4. What does the vector field look like along the line $x = 1$ and along the line $x = -1$? What can you say about the special behavior of solutions with an initial condition $(\pm 1, y_0)$? Interpret this geometrically.

Problem 5. What does the vector field look like along the line $y = 1$ and along the line $y = -1$? What can you say about the special behavior of solutions with an initial condition $(x_0, \pm 1)$? Interpret this geometrically.

Let $A \in M_2(\mathbb{R})$. Let τ be the trace of A, and let δ be the determinant of A. The characteristic polynomial for A will factor as

$$\begin{aligned} p(x) &= (\lambda_1 - x)(\lambda_2 - x) \\ &= x^2 - (\lambda_1 + \lambda_2)x + \lambda_1\lambda_2 \\ &= x^2 - \tau x + \delta \end{aligned}$$

where λ_1 and λ_2 are the eigenvalues of A. Setting $p(x) = 0$ and solving gives an alternate description of the eigenvalues:

$$x = \frac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2}.$$

If $\delta = 0$, then at least one of the eigenvalues is zero, and we have a **degenerate system**.

$\boxed{\delta = 0}$ **degenerate**.

$\boxed{\delta < 0}$ real eigenvalues, opposite signs $\Rightarrow$ **saddle**.

$\boxed{\delta > 0,\ \tau^2 - 4\delta \geq 0}$ real eigenvectors, same signs $\Rightarrow$ node.

$\tau < 0 \Rightarrow$ **stable node**

$\tau > 0 \Rightarrow$ **unstable node**.

$\boxed{\delta > 0,\ \tau^2 - 4\delta < 0}$ nonreal eigenvectors $\Rightarrow$ swirling vector field.

$\tau < 0 \Rightarrow$ **stable focus**

$\tau > 0 \Rightarrow$ **unstable focus**

$\tau = 0 \Rightarrow$ **center**.