Math 322 worksheet for Wednesday, Week 7

Definition. An *equilibrium point* for a system of differential equations in $\mathbb{R}^n$

$$x' = f(x)$$

is a point $p \in \mathbb{R}^n$ such that $f(p) = 0$.

The reason for the terminology is that if p is an equilibrium point then a solution (*the solution* if f is continuously differentiable) with initial condition $x(0) = p$ is the constant solution $x(t) = p$.

We hope to get a qualitative sense of the solutions to our system near an equilibrium point p by replacing the system with a linear approximation:

$$x' = Jf_p$$

where Jf_p is the Jacobian matrix for f at p.

Consider the system of equations

$$\begin{aligned} x' &= (x^2 - 1)y \\ y' &= (1 - y^2)\left(x + \frac{3}{10}y\right). \end{aligned}$$

So in this case $f(x, y) = ((x^2 - 1)y, (1 - y^2)\left(x + \frac{3}{10}y\right))$.

Problem 1. Find all equilibrium points for the system and plot them in the plane.

SOLUTION: We need to solve the system

$$\begin{aligned} (x^2 - 1)y &= 0 \\ (1 - y^2)\left(x + \frac{3}{10}y\right) &= 0. \end{aligned}$$

The top equation is satisfied if and only if $x = \pm 1$ or $y = 0$. Consider these two cases separately. If $x = \pm 1$, then the second equation is satisfied if and only if $y = \pm 1$ or $y = \pm 10/3$ (the latter depending on the sign of x). Thus, in the first case, we find the equilibrium points

$$(\pm 1, \pm 1),\ \left(1, -\frac{10}{3}\right),\ \left(-1, \frac{10}{3}\right).$$

Next, consider the case where $y = 0$. The second equation then gives $x = 0$. So we get a seventh equilibrium point at the origin: $(0, 0)$.

Problem 2. Compute the Jacobian matrix $Jf_{(x,y)}$ for our f at an arbitrary point (x, y).

SOLUTION: We have

$$f(x,y) = \left((x^2-1)y, (1-y^2)\left(x + \frac{3}{10}y\right)\right)$$

So

$$Jf_{(x,y)} = \begin{pmatrix} 2xy & x^2-1 \\ 1-y^2 & -2y\left(x+\frac{3}{10}y\right)+\frac{3}{10}(1-y^2) \end{pmatrix}.$$

Problem 3. For each equilibrium point p, analyze the linear system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = Jf_p \begin{pmatrix} x \\ y \end{pmatrix}$$

by looking at the eigenvalues of Jf_p. Do you get a saddle? A stable focus or node? An unstable focus or node? A center? (See the last page for a quick guide.)

SOLUTION:

$\boxed{(0,0)}$

$$Jf_{(0,0)} = \begin{pmatrix} 0 & -1 \\ 1 & \frac{3}{10} \end{pmatrix}.$$

The trace is $\tau = \frac{3}{10}$ and the determinant is $\delta = 1$. So $\delta > 0$, $\tau > 0$, and

$$\tau^2 - 4\delta = \frac{9}{100} - 4 < 0.$$

This means the origin is an **unstable focus**. (The characteristic polynomial has two unreal eigenvalues and the real parts of the eigenvalues are positive.)

$\boxed{(1,1)}$

$$Jf_{(1,1)} = \begin{pmatrix} 2 & 0 \\ 0 & -\frac{13}{5} \end{pmatrix}.$$

The determinant is $\delta < 0$. This means we have a **saddle**.

$\boxed{(1,-1)}$

$$Jf_{(1,-1)} = \begin{pmatrix} -2 & 0 \\ 0 & \frac{7}{5} \end{pmatrix}.$$

The determinant is $\delta < 0$, another **saddle**.

$\boxed{(-1,1)}$

$$Jf_{(-1,1)} = \begin{pmatrix} -2 & 0 \\ 0 & \frac{7}{5} \end{pmatrix}.$$

The determinant is $\delta < 0$, another **saddle**.

$\boxed{(-1,-1)}$

$$Jf_{(-1,-1)} = \begin{pmatrix} 2 & 0 \\ 0 & -\frac{13}{10} \end{pmatrix}.$$

The determinant is $\delta < 0$, another **saddle**.

$\boxed{(1,-\frac{10}{3})}$

$$Jf_{(1,-10/3)} = \begin{pmatrix} -\frac{20}{3} & 0 \\ -\frac{91}{9} & -\frac{91}{30} \end{pmatrix}.$$

The determinant is $\delta > 0$, $\tau < 0$, and

$$\tau^2 - 4\delta = \left(-\frac{97}{10}\right)^2 - 4\left(-\frac{20}{3}\right)\left(-\frac{91}{30}\right) = -8153/90 < 0.$$

Therefore, we get a **stable node**: two real eigenvalues, both negative.

$\boxed{(-1,\frac{10}{3})}$

$$Jf_{(-1,10/3)} = \begin{pmatrix} -\frac{20}{3} & 0 \\ -\frac{91}{9} & -\frac{97}{30} \end{pmatrix},$$

which is the same matrix we had at $(1,-\frac{10}{3})$. Therefore, we get another **stable node**: two real eigenvalues, both negative.

Problem 4. What does the vector field look like along the line $x = 1$ and along the line $x = -1$? What can you say about the special behavior of solutions with an initial condition $(\pm 1, y_0)$? Interpret this geometrically.

SOLUTION: The vector field along the line $x = 1$ is

$$f(1,y) = \left(0, (1-y^2)\left(1+\frac{3}{10}y\right)\right).$$

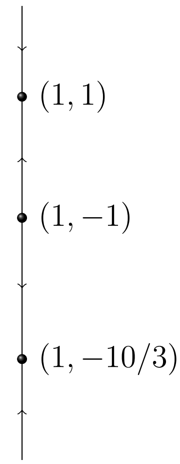
The second coordinate is 0 if $y = \pm 1$ or if $y = -10/3$. So we have the following cases:

$$y \in (1, \infty) \Rightarrow f(1, y)_2 < 0$$

$$y \in (-1, 1) \Rightarrow f(1, y)_2 > 0$$

$$y \in (-10/3, -1) \Rightarrow f(1, y)_2 < 0$$

$$y \in (-\infty, -10/3) \Rightarrow f(1, y)_2 > 0$$

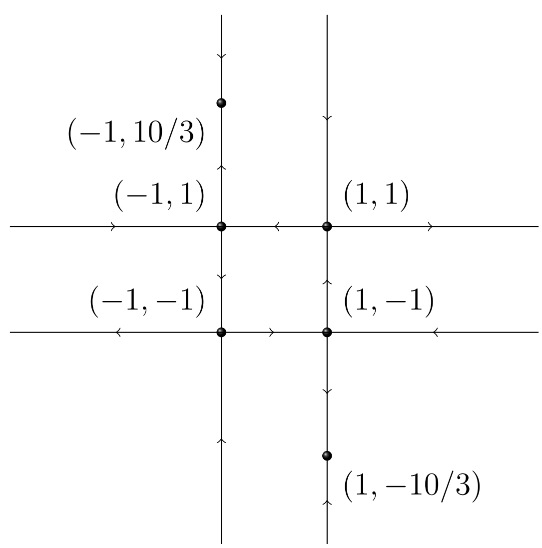


Trajectories starting at a point on the line $x = 1$ at a point below $y = -10/3$ or between $y = -10/3$ and $y = -1$ get sucked into the equilibrium point $(1, -10/3)$. Trajectories starting at a point on the line $x = 1$ at a point above $y = -1$ get sucked into the equilibrium point at $(1, 1)$.

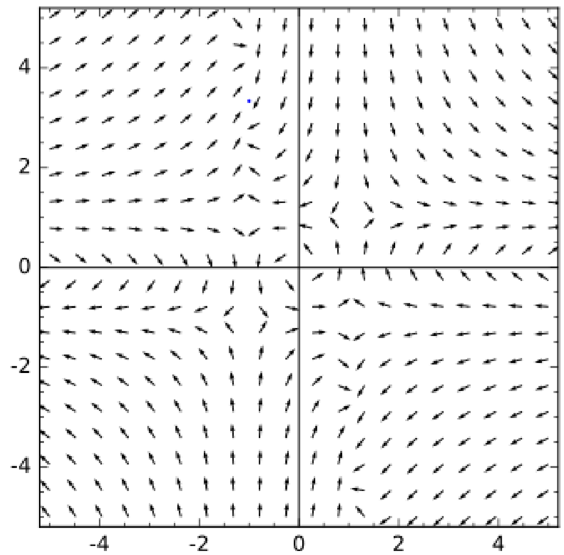
The case for $x = -1$ is similar. We show the picture for $x = \pm 1$ and $y = \pm 1$ in the solution to problem 5, below.

Problem 5. What does the vector field look like along the line $y = 1$ and along the line $y = -1$? What can you say about the special behavior of solutions with an initial condition $(x_0, \pm 1)$? Interpret this geometrically.

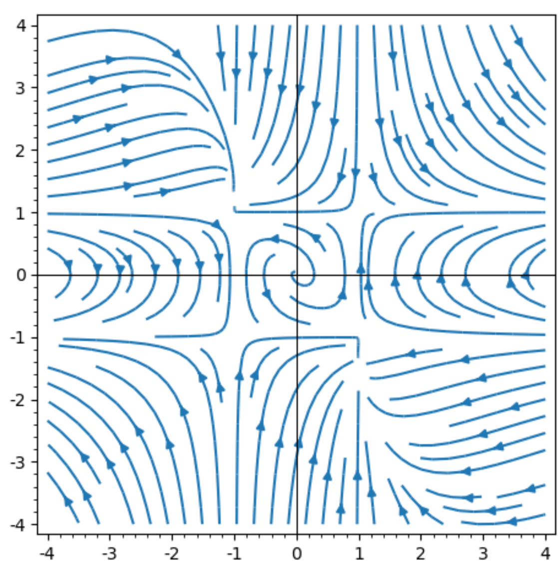
SOLUTION:



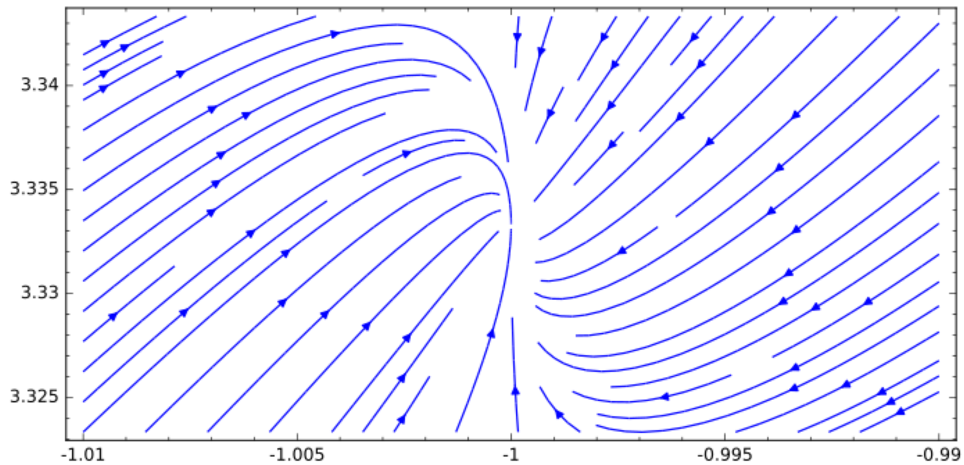
Here is a picture of the vector field (normalized so that each arrow has the same length):



Here is a picture of the flow of the vector field:



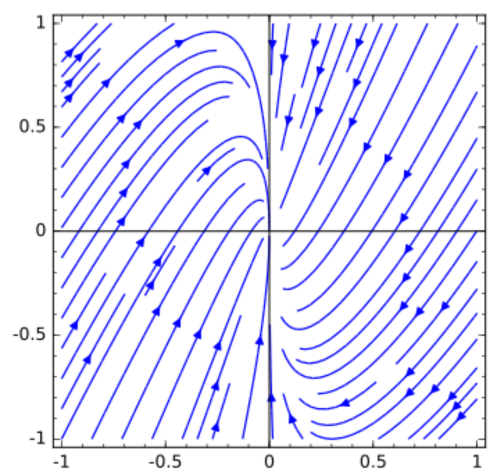
The vector fields near the equilibrium points $(-1, 10/3)$ and $(1, -10/3)$ both look like this:



The linearizations at both $(-1, 10/3)$ and $(1, -10/3)$ are the same:

$$x' = -\frac{20}{3}x$$
$$y' = -\frac{91}{9}x - \frac{91}{30}y,$$

and the vector field looks like:



Let $A \in M_2(\mathbb{R})$. Let τ be the trace of A, and let δ be the determinant of A. The characteristic polynomial for A will factor as

$$\begin{aligned} p(x) &= (\lambda_1 - x)(\lambda_2 - x) \\ &= x^2 - (\lambda_1 + \lambda_2)x + \lambda_1\lambda_2 \\ &= x^2 - \tau x + \delta \end{aligned}$$

where λ_1 and λ_2 are the eigenvalues of A. Setting $p(x) = 0$ and solving gives an alternate description of the eigenvalues:

$$x = \frac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2}.$$

If $\delta = 0$, then at least one of the eigenvalues is zero, and we have a **degenerate system**.

$\boxed{\delta = 0}$ **degenerate**.

$\boxed{\delta < 0}$ real eigenvalues, opposite signs $\Rightarrow$ **saddle**.

$\boxed{\delta > 0,\ \tau^2 - 4\delta \geq 0}$ real eigenvectors, same signs $\Rightarrow$ node.

$\tau < 0 \Rightarrow$ **stable node**
$\tau > 0 \Rightarrow$ **unstable node**.

$\boxed{\delta > 0,\ \tau^2 - 4\delta < 0}$ nonreal eigenvectors $\Rightarrow$ swirling vector field.

$\tau < 0 \Rightarrow$ **stable focus**
$\tau > 0 \Rightarrow$ **unstable focus**
$\tau = 0 \Rightarrow$ **center**.