Math 322 lecture for Monday, Week 7

FUNDAMENTAL EXISTENCE AND UNIQUENESS THEOREM

Our goal is to apply the contraction mapping principle to the operator

$$\begin{aligned} T\colon C(I) &\to C(I) \\ u &\mapsto x_0 + \int_{s=0}^{t} f(u(s))\,ds \end{aligned}$$

in order to prove the fundamental existence and uniqueness theorem for ordinary differential equations.

Derivative review. Let $E \subseteq \mathbb{R}^n$ be an open set. Recall from vector calculus that the derivative of a function $f\colon E \to \mathbb{R}^n$ at a point $p \in E$ is a linear function

$$Df_p\colon \mathbb{R}^n \to \mathbb{R}^n$$

approximating f near p:

$$f(p+h) \approx f(p) + Df_p(h)$$

for small h. Its corresponding matrix is the Jacobian matrix for f at p, whose j-th column is the j-th partial of f (measuring how f is changing in the j-th coordinate direction):

$$\frac{\partial f}{\partial x_j}(p) = \begin{pmatrix} \frac{\partial f_1}{\partial x_j}(p) \\ \frac{\partial f_2}{\partial x_j}(p) \\ \vdots \\ \frac{\partial f_n}{\partial x_j}(p) \end{pmatrix}.$$

We say $f\colon E \to \mathbb{R}^n$ is *continuously differentiable* if it is differentiable at all points in E and the mapping

$$\begin{aligned} E &\to \mathcal{L}(\mathbb{R}^n) \\ p &\mapsto Df_p \end{aligned}$$

is continuous.

Explanation: First, $\mathcal{L}(\mathbb{R}^n)$ denotes the vector space of linear functions from $\mathbb{R}^n$ to itself. Second, to talk about continuity we define a norm on $\mathcal{L}(\mathbb{R}^n)$: for $L \in \mathcal{L}(\mathbb{R}^n)$, let

$$\|L\| = \max_{|x|\le 1} |L(x)|.$$

This is the same as $\|A\|$ if A is the matrix representing L. In that case, since $L(x) = Ax$, the inequality $|Ax| \leq \|A\||x|$ can be written as

$$\|L(x)\| \leq \|L\| \, |x|.$$

A theorem from calculus says that f is continuously differentiable if and only if all of its partials $\partial f_i/\partial x_j$ exist and are continuous. (Also, it turns out that continuity of the partials guarantees that f is differentiable.)

Notation. For an open subset $E \subset \mathbb{R}^n$, we denote the $\mathbb{R}$-vector space of continuously differentiable functions on E by $C^1(E)$.

Lipschitz condition. We now introduce a condition on vector fields that will allow the application of the contraction mapping principle to T.

Definition. Let $E \subseteq \mathbb{R}^n$ be an open subset. Then a function $f\colon E \to \mathbb{R}^n$ is *Lipschitz* if there exists a constant K such that

$$|f(x) - f(y)| \leq K|x - y|$$

for all $x, y \in E$. On the other hand, f is *locally Lipschitz* on E if for each $x_0 \in E$, there exists $\varepsilon > 0$ and a constant K_{x_0} such that

$$|f(x) - f(y)| \leq K_{x_0}|x - y|$$

for all

$$x, y \in N_\varepsilon(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < \varepsilon\}.$$

Proposition. If $f \in C^1(E)$, then f is locally Lipschitz.

Proof. Let $x_0 \in E$. Since E is open it contains an open ball about x_0, i.e., there exists $\eta > 0$ such that $N_\eta(x_0) \subset E$. Define $\varepsilon := \eta/2$ and consider the closed ball

$$B := B_\varepsilon(x_0) := \overline{N_\varepsilon(x_0)} := \{x \in \mathbb{R}^n : |x - x_0| \leq \varepsilon\}.$$

Let

$$K_{x_0} := \max_{x \in B} \|Df_x\|.$$

The constant K_{x_0} exists since we're assuming Df is continuous ($f \in C^1(E)$). Thus, $x \to Df_x \to \|Df_x\|$, being the composition of continuous functions, is also continuous.

Since B is convex, given $x, y \in B$, the line segment joining x to y is contained in B. Hence, it is OK to stick these points into f. Parametrize the line segment by $\phi(s) = x + s(y - x)$ for $s \in [0, 1]$ and consider the composition

$$\begin{aligned} F := f \circ \phi :: [0,1] &\to \mathbb{R}^n \\ s &\mapsto f(x + s(y - x)), \end{aligned}$$

a curve in $\mathbb{R}^n$. By the chain rule,

$$DF_s = Df_{\phi(s)} \circ D\phi_s.$$

Since F is a curve in $\mathbb{R}^n$, its Jacobian matrix at s is a single column vector—the tangent or velocity vector $F'(s)$—and

$$DF_s(t) = tF'(s),$$

a linear function of t (for fixed s). Similarly ϕ_s is a curve in $\mathbb{R}^n$, so its Jacobian matrix is its velocity at time s. It's easy to compute: since $\phi(s) = x + s(y - x)$, its velocity is constant. At any time s, we have $\phi'(s) = y - x$. Thus,

$$D\phi_s(t) = t(y - x).$$

By the chain rule,

$$tF'(s) = DF_s(t) = Df_{\phi(s)}(t(y - x)).$$

Setting $t = 1$, we get

$$F'(s) = Df_{(x+s(y-x))}(y - x) \in \mathbb{R}^n.$$

Since $F(0) = f(x)$ and $F(1) = f(y)$,

$$\begin{aligned} |f(y) - f(x)| &= |F(1) - F(0)| \\ &= \left| \int_{s=0}^{1} F'(s)\, ds \right| \\ &\le \int_{s=0}^{1} |F'(s)|\, ds \\ &= \int_{s=0}^{1} |Df_{(x+s(y-x))}(y - x)|\, ds \\ &\le \int_{s=0}^{1} \|Df(x + s(y - x))\|\, |y - x|\, ds \\ &\le K_{x_0} \int_{s=0}^{1} |y - x|\, ds \\ &= K_{x_0}|y - x|. \end{aligned}$$

We've shown that f is locally Lipschitz. $\square$

Theorem. (The fundamental existence and uniqueness theorem for nonlinear systems.) Let E be an open subset of $\mathbb{R}^n$ containing x_0, and let $f \in C^1(E)$. Then there exists $a > 0$ such that the initial value problem

$$\begin{aligned} x' &= f(x) \\ x(0) &= x_0 \end{aligned}$$

has a unique solution $x(t)$ on $[-a, a]$.

Proof. Since $f \in C^1(E)$, there exists an $\varepsilon > 0$ such that $N_\varepsilon(x_0) \subseteq E$, the open ball of radius ε centered at x_0, and there exists a constant K_{x_0} such that

$$|f(x) - f(y)| \leq K_{x_0}|x - y|$$

for all x, y in $N_\varepsilon(x_0)$. By replacing ε by $\varepsilon/2$, we may assume

$$|f(x) - f(y)| \leq K_{x_0}|x - y|$$

for all x, y in

$$B := \overline{N_\varepsilon(x_0)} := \{x \in \mathbb{R}^n : |x - x_0| \leq \varepsilon\} \subset E.$$

(The point here is to get the Lipschitz condition to hold on a closed bounded ball rather then on the open ball, $N_\varepsilon(x_0)$, in preparation for an application of the extreme value theorem, below.)

Let $I = [-a, a]$ where $a > 0$ is a constant to be determined later, and define

$$X := \{u \in C(I)\colon \|u - x_0\| \leq \varepsilon\},$$

considering $x_0 \in C(I)$ as the constant function $t \mapsto x_0$ for all $t \in I$. This means that for $u \in X$, we have

$$\max_{t \in I} |u(t) - x_0| \leq \varepsilon.$$

In particular, $u(t) \in B \subset E$ for all $t \in I$. Note that B is a subset of $E \subseteq \mathbb{R}^n$ and X is a subset of the function space $C(I)$ of continuous functions $I \to \mathbb{R}^n$. If $u \in X$, then $u(t) \in B$ for all $t \in I$.

Our goal is to show that a can be taken small enough so that (i) $T(u) \in X$ for all $u \in X$, i.e., so that $T\colon X \to X$, and so that (ii) $T\colon X \to X$ is a contraction mapping.

For (i), since B is closed and bounded, we can define

$$M = \max_{x \in B} |f(x)|.$$

Suppose that $0 < a < \frac{\varepsilon}{M}$. Then for $u \in X$ and $t \in I$,

$$\begin{aligned} |T(u)(t) - x_0| &= \left| \left(x_0 + \int_{s=0}^{t} f(u(s))\, ds \right) - x_0 \right| \\ &= \left| \int_{s=0}^{t} f(u(s))\, ds \right| \\ &\leq \left| \int_{s=0}^{t} |f(u(s))|\ ds \right|. \end{aligned}$$

If s is in the interval between 0 and t and $u \in X$, it follows that $u(s) \in B$, and hence, $|f(u(s))| \leq M$. Therefore, continuing our calculation,

$$\begin{aligned} |T(u)(t) - x_0| &= \left| \int_{s=0}^{t} |f(u(s))|\ ds \right| \\ &= \left| \int_{s=0}^{t} M\, ds \right| \\ &= |t|\, M \\ &\leq a\, M \\ &< \frac{\varepsilon}{M} M \\ &< \varepsilon. \end{aligned}$$

Hence,

$$\|T(u) - x_0\| := \max_{t \in I} |T(u)(t) - x_0| < \varepsilon.$$

Therefore $T(u) \in X$. In sum: if $0 < a < \varepsilon/M$, then $T\colon X \to X$.

We now work on (ii): we can take a small enough so that $T\colon X \to X$ is a contraction mapping. Let $u, v \in X$. Then, using the Lipschitz property,

$$\begin{aligned} |T(u) - T(v)| &= \left| \int_{s=0}^{t} f(u(s)) - f(v(s))\, ds \right| \\ &\leq \left| \int_{s=0}^{t} |f(u(s)) - f(v(s))|\ ds \right| \end{aligned}$$

$$
\begin{aligned}
&\leq K_{x_0} \left| \int_{s=0}^{t} |u(s) - v(s)| \, ds \right| \\
&\leq K_{x_0} \left| \int_{s=0}^{t} \max_{c \in I} |u(c) - v(c)| \, ds \right| \\
&= K_{x_0} \left| \int_{s=0}^{t} \|u - v\| \, ds \right| \\
&= K_{x_0} \, |t| \, \|u - v\| \\
&\leq a K_{x_0} \|u - v\|.
\end{aligned}
$$

To ensure T is a contraction mapping, take $a = \frac{1}{2K_{x_0}}$ (so that $aK_{x_0} = \frac{1}{2} < 1$).

In total, we have now shown there exists and interval $I = [-a, a]$, a closed ball $X \subset C(I)$ centered at the constant function x_0, such that $T \colon X \to X$ and T is a contraction mapping. It therefore has a unique fixed point $x \in X$. So $x = T(x)$, i.e.,

$$
x(t) = T(x)(t) := x_0 + \int_{s=0}^{t} f(x(s)) \, ds.
$$

By the fundamental theorem of calculus and the fact that $x(0) = x_0$, it follows that x is a solution to the initial value problem

$$
\begin{aligned}
x' &= f(x) \\
x(0) &= x_0
\end{aligned}
$$

on I. For uniqueness, recall that any solution x on I will be a fixed point for T:

$$
T'(x)(t) = \left(x_0 + \int_{s=0}^{t} f(x(s)) \, ds \right)' = f(x(t)) = x'(t).
$$

so $T(x)$ and x differ by a constant. However $T(x(0)) = x_0 = x(0)$, so that constant is 0. Since every solution is a fixed point of T and contraction mappings have unique fixed points, we are done. $\square$