Math 322 lecture for Friday, Week 7

Dependence on parameters, maximal interval

Here we mention a couple of fairly immediate refinements of the fundamental existence and uniqueness theorem. Consider our usual initial value problem:

$$\begin{aligned} x' &= f(x) \\ x(0) &= x_0 \end{aligned} \tag{1}$$

where $f\colon E \to \mathbb{R}^n$ is continuously differentiable on the open subset $E \subset \mathbb{R}^n$ and $x_0 \in E$. The first refinement (dependence on parameters) says that if we deform f smoothly and move x_0 slightly, then the solution deforms smoothly. The second refinement says that the solution $x(t)$ to our initial value problem exists on a uniquely determined maximal interval about $t = 0$.

Theorem. (Dependence on parameters.) Let E be an open subset of $\mathbb{R}^{n+m}$ containing the point (x_0, μ_0) where $x_0 \in \mathbb{R}^n$ and $\mu_0 \in \mathbb{R}^m$, and assume $f \in C^1(E)$. Then there is a neighborhood[1] $N(x_0) \subseteq \mathbb{R}^n$ of x_0, a neighborhood $N(\mu_0) \subseteq \mathbb{R}^m$ of μ_0, and an $a > 0$ such that for all $y \in N(x_0)$ and for all $\mu \in N(\mu_0)$, the initial value problem

$$\begin{aligned} x' &= f(x, \mu) \\ x(0) &= y \end{aligned}$$

has a unique solution $x = x(t, y, \mu)$ with $x \in C^1(R)$ where $R := [-a, a] \times N(x_0) \times N(\mu_0)$.

Example. Let $A \in M_n(\mathbb{R})$ and $x_0 \in \mathbb{R}^n$. Then the solution to the system $x' = Ax$ with $x(0) = x_0$ is $x(t, x_0, A) = e^{At}x_0$, which is a smooth function of t, A, and x_0. In this case, $m = \binom{n}{2}$, and we identify a point $\mu \in \mathbb{R}^m$ with a matrix A_μ whose entries, read from left-to-right, top-to-bottom form μ. Thus, $f(x, \mu) = A_\mu x$.

Theorem. Consider our initial value problem with $f \in C^1(E)$ and initial condition x_0. There is an interval $J = (\alpha, \beta)$ with $\alpha, \beta \in \mathbb{R} \cup \{\pm\infty\}$ and a solution $x(t)$ defined for $t \in J$ such that if $y(t)$ is any other solution defined on an interval I, then $I \subseteq J$ and $x(t) = y(t)$ on I. Further, if $\beta \in \mathbb{R}$, i.e., if $\beta \neq \infty$, then given any compact (closed and bounded) subset $K \subset E$, then there exists $t \in J$ such that $x(t) \notin K$.

The interval J is called the *maximal interval of existence* and is clearly uniquely determined.

[1] A *neighborhood* of a point is any set that contains an open set containing the point.

Last lecture, we started investigating the effect of replacing $f(x)$ with Df_{x_0} in (1) at an equilibrium point x_0, i.e., at a point where $f(x_0) = 0$. The first theorem we'll consider which makes this comparison precise is the *stable manifold theorem.* To state the theorem we need to formally introduce the *flow* of a vector field, and the idea of a *manifold.*

Flow. For each $x_0 \in E$, let $I(x_0)$ be the maximal interval of existence of the solution to (1) with initial condition x_0. Then let

$$\Omega := \{(t, x_0) \in \mathbb{R} \times E : t \in I(x_0)\}.$$

For each $(t, x_0) \in \Omega$, let $\phi(t, x_0)$ be the solution to (1) with initial condition x_0 evaluated at time $t \in I(x_0)$. This defines a mapping

$$\phi \colon \Omega \to \mathbb{R}^n$$

called the *flow* of the vector field $f \colon E \to \mathbb{R}^n$. For each $t \in I(x_0)$ we define

$$\phi_t(x_0) := \phi(t, x_0).$$

Our text (Section 2.5) establishes the following properties for the flow:

(a) $\phi_0(x_0) = x_0$

(b) $\phi_s(\phi_t(x_0)) = \phi_{s+t}(x_0)$

(c) $\phi_{-t}(\phi_t(x_0)) = x_0$

wherever these expressions make sense.

Example. Consider the case of a linear system, in which $f(x) = Ax$ for some $A \in M_n(\mathbb{R})$. Here $E = \mathbb{R}^n$, and for each $x_0 \in \mathbb{R}^n$, the solution is

$$\phi_t(x_0) = x(t) = e^{At}x_0,$$

and the maximal interval of existence is $I(x_0) = \mathbb{R}$. So $\Omega = \mathbb{R}^{n+1}$, and the above properties for the flow are easily verified in this special case. For instance,

$$\phi_s(\phi_t(x_0)) = e^{As}(e^{At}x_0) = e^{A(s+t)}x_0 = \phi_{s+t}(x_0).$$

Manifolds. Roughly speaking, a manifold is a object that can be constructed from a collection of open subsets of $\mathbb{R}^n$ and a set of instructions for gluing these open sets together. A quintessential example is given by an ordinary world atlas. Each page consists of a flattened out map of a piece of the earth. There will be pairs of pages that overlap along boundaries representing the same regions. The drawings of features of the earth on these pages implicitly provide instructions for gluing the pages together. If the pages where made of moldable putty, then it would be possible to piece these pages together to make a shape. One possible result, among others would be a sphere, and so we say the sphere is a manifold. It is two-dimensional since we glue together open subsets of $\mathbb{R}^2$ to make it. We now move on to the formal definition.

Definition. A *metric space* is a set X with a *distance function* or *metric*,

$$d\colon X \times X \to \mathbb{R}$$

that is positive definite, symmetric, and obeys the triangle inequality:

(a) $d(x, y) \geq 0$ with $d(x, y) = 0$ if and only if $x = y$

(b) $d(x, y) = d(y, x)$

(c) $d(x, y) \leq d(x, z) + d(z, y)$.

Every metric space (X, d) is a topological space where a subset $U \subseteq X$ is *open* if for each $u \in U$, there exists $r > 0$ such that the open ball of radius r centered at u is contained in U:

$$B(u, r) := \{x \in X : d(u, x) < r\} \subseteq U.$$

Definition. Two subsets A, B of a metric space X are *homeomorphic* if there exists a continuous bijection $f\colon A \to B$ with continuous inverse. The mapping f is then called a *homeomorphism* from A to B. (More generally, two topological spaces U, V are homeomorphic if there is a continuous bijection $f\colon U \to V$ with continuous inverse.)

Definition. An n-dimensional differentiable manifold is a connected metric space[2] M and an open covering $\{U_\alpha\}$ (so for each α in some index set, U_α is an open subset of M and $M = \cup_\alpha U_\alpha$) such that:

[2]More generally, M could be a second-countable Hausdorff toplogical space.

(a) for all α, there is a homeomorphism

$$h_\alpha \colon U_\alpha \to V_\alpha$$

where V_α is an open subset of $\mathbb{R}^n$, and

(b) if $U_\alpha \cap U_\beta \neq \emptyset$, the mapping

$$h_\beta \circ h_\alpha^{-1} \colon h_\alpha(U_\alpha \cap U_\beta) \to h_\beta(U_\alpha \cap U_\beta)$$

is continuously differentiable.

Each pair (h_α, U_α) is called a *chart*, and the collection of charts is called an *atlas*. The mapping $h_\beta \cap h_\alpha^{-1}$ are *transition functions*.

To go back to the rough description we made earlier: each chart (h_α, U_α) represents a page $h_\alpha(U_\alpha)$ in the atlas. The set U_α is a piece of the manifold (earth), and the mapping h_α is the rendering of that piece of the earth onto a flat piece of paper. On overlaps $U_\alpha \cap U_\beta$ on the manifold the corresponding pages of the atlas have overlaps $h_\alpha(U_\alpha \cap U_\beta)$ and $h_\beta(U_\alpha \cap U_\beta)$. We can glue these together with the transition function $h_\beta \circ h_\alpha^{-1}$.

Theorem. (Stable manifold theorem.) Let $E \subseteq \mathbb{R}^n$ and let $f \in C^1(E)$. Suppose that $f(0) = 0$ and that Df_0 has k eigenvalues with negative real part and $n - k$ eigenvalues with positive real part. Then there exists a k-dimensional differentiable manifold S tangent to the stable subspace E^s of the linearized system $x' = Df_0(x)$ at 0 and there exists an $(n-k)$-dimensional differentiable manifold U tangent to the unstable space E^u of the linearized system. Further

$$\lim_{t \to \infty} \phi_t(x_0) = 0$$

for any $x_0 \in S$ and

$$\lim_{t \to -\infty} \phi(x_0) = 0$$

for any $x_0 \in U$.