Math 322 lecture for Monday, Week 6

n-TH ORDER LINEAR HOMOGENEOUS EQUATIONS REVISITED

Consider the homogeneous linear equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \tag{1}$$

with initial condition $y^{(i)}(0) = b_i$ for $i = 0, 1, \dots, n-1$. Recall the method of solution introduced during the first two weeks of class. First we factor the characteristic polynomial

$$P(x) = \prod_{i=1}^{k} (x - \lambda_i)^{m_i},$$

where the λ_i are distinct. We claimed that the most general solution was an arbitrary linear combination of the basic functions

$$e^{\lambda_i t},\ te^{\lambda_i t},\ \dots,\ t^{m_i - 1}e^{\lambda_i t}$$

for $i = 1, \dots, k$. We also claimed that for each initial condition, there would be a unique solution. We now want to justify those claims.

Define

$$x_1 := y,\ x_2 := y',\ x_3 := y'',\ \dots\ ,\ x_n := y^{(n-1)}.$$

We get a corresponding matrix equation

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ \vdots \\ x_{n-1}' \\ x_n' \end{pmatrix} = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & \text{\huge 0} & \\ & & 0 & 1 & & \\ \text{\huge 0} & & & \ddots & \ddots & \\ & & & & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-2} & -a_{n-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}.$$

Let A denote the $n \times n$ matrix above, and we can consider the differential equation $x' = Ax$ with initial condition $x_0 = x(0) = (y(0), y'(0), \dots, y^{(n-1)}(0)) = (b_0, b_1, \dots, b_{n-1})$. The solution is

$$x(t) = e^{At}x_0,$$

and the first component of this solution is $x_1(t) = y(t)$, the solution to the original homogeneous system.

Proposition. Let $P(x) = \sum_{i=0}^{n} a_i x^i$ be the characteristic polynomial for the linear homogeneous equation (1). Then the characteristic polynomial for A is

$$p_A(x) := \det(A - xI_n) = (-1)^{n+1} P(x).$$

Proof. We have

$$A - xI_nA = \begin{pmatrix} -x & 1 & & & & \\ & -x & 1 & & \text{\huge 0} & \\ & & -x & 1 & & \\ & \text{\huge 0} & & \ddots & \ddots & \\ & & & & -x & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-2} & -x - a_{n-1} \end{pmatrix}$$

Multiply the second column by x and add it to the first column; then multiply the third column by x^2 and add it to the first column; and so on. This does not affect the determinant, and the rows of the first column are all 0 now except the last, which is

$$-a_0 - a_1 x - a_2 x^2 - \cdots - a_{n-2} x^{n-2} - (x + a_{n-1}) x^{n-1} = -P(x).$$

It follows that

$$\det(A - xI_n) = \det \begin{pmatrix} 0 & 1 & & & & \\ 0 & -x & 1 & & \text{\huge 0} & \\ 0 & & -x & 1 & & \\ \vdots & \text{\huge 0} & & \ddots & \ddots & \\ 0 & & & & -x & 1 \\ -P(x) & -a_1 & -a_2 & \dots & -a_{n-2} & -a_{n-1} - x \end{pmatrix}$$

Expand along the first column to get the result. $\square$

In order to solve the system, we are interested in the Jordan form for A. So we think about this next.

Proposition. Let λ be an eigenvalue for A. Then the corresponding eigenspace is

$$E_\lambda = \operatorname{Span}\{(1, \lambda, \lambda^2, \dots, \lambda^{n-1})\}$$

and is, hence, one-dimensional. So the geometric multiplicity of each eigenvalue for A is 1.

Proof. Suppose that $Av = \lambda v$ where $v = (v_1, \ldots, v_n)$. Note that

$$Av = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & \ddots & \ddots & \\ & & & & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-2} & -a_{n-1} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{n-1} \\ v_n \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{n-1} \\ v_n \end{pmatrix} = \lambda v$$

says $v_2 = \lambda v_1,\ v_3 = \lambda v_2,\ \ldots\ ,\ v_n = \lambda v_{n-1}$. Thus,

$$\begin{aligned} v &= (v_1, \lambda v_1, \lambda^2 v_1, \ldots, \lambda^{n-1} v_1) \\ &= v_1(1, \lambda, \lambda^2, \ldots, \lambda^{n-1}). \end{aligned}$$

□

Corollary. Suppose that A has distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ (over $\mathbb{C}$) with algebraic multiplicities, $m_1, \ldots, m_k$, respectively, so the its characteristic polynomial is

$$p_A(x) = \prod_{i=1}^{k} (\lambda_i - x)^{m_i}.$$

Then the Jordan form for A is

$$\begin{pmatrix} J_{m_1}(\lambda_1) & & & \\ & J_{m_2}(\lambda_2) & & \text{\huge 0} \\ & & \ddots & \\ & \text{\huge 0} & & J_{m_k}(\lambda_k) \end{pmatrix}.$$

Proof. This follows immediately from the preceding Proposition. The diagonal of the Jordan form consists of the eigenvalues of A, repeated according to multiplicities. For each Jordan block, there is a corresponding eigenvector for A (and several generalized eigenvectors). If there where more than one Jordan block for a particular eigenvalue λ, there would be more than one linearly independent eigenvector for λ, and we've just seen that this cannot happen—each eigenspace has dimension 1. □

Theorem. Suppose the roots for the characteristic polynomial for equation (1) or, equivalently, the eigenvalues for A are $\lambda_1, \ldots, \lambda_k$ with multiplicities $m_1, \ldots, m_k$, respectively. Every solution to equation (1) (with a given initial condition) is a unique linear combination of the *basic functions*

$$\left\{ t^j e^{\lambda_i t} : 0 \leq j < m_i, 1 \leq i \leq k \right\}, \tag{2}$$

and each linear combination of these functions is a solution for some initial condition.

Proof. There are three parts to this proof: (i) show each solution is a linear combination of the basic functions; (ii) show each basic function satisfies the differential equation (1); and (iii) show the basic equations are linearly independent.
(i) The solution to equation (1) is the first component of $e^{At}x_0$. Letting $P^{-1}AP = J$ be the Jordan form for A, the solution is

$$y(t) = e^{At}x_0 = Pe^{Jt}P^{-1},$$

and hence, is a linear combination of the entries of e^{Jt}. The result then follows from the previous corollary recalling that

$$e^{J_{m_i}(\lambda_i t)} = e^{\lambda_i t}\begin{pmatrix} 1 & t & \frac{t^2}{2!} & \dots & \dots & \frac{t^{m_i-1}}{(m_i-1)!} \\ 0 & 1 & t & \dots & \dots & \frac{t^{k-2}}{(m_i-2)!} \\ 0 & 0 & 1 & \dots & \dots & \frac{t^{k-3}}{(m_i-3)!} \\ & \ddots & & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 & 1 & t \\ 0 & \dots & \dots & \dots & 0 & 1 \end{pmatrix}.$$

(ii) Consider the differential operator $D := \frac{d}{dt}$. We can write equation (1) as

$$P(D)y = 0$$

where $P(D) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$. We are given that

$$P(D) = \prod_{i=1}^{k}(D - \lambda_i)^{m_i}.$$

That the basic functions satisfy the differential equation $P(D)y = 0$ is left as homework. It follows from two facts (which are part of the homework problem):

(a) $(D - \alpha)(D - \beta)f(t) = (D - \beta)(D - \alpha)f(t)$ for every sufficiently differentiable function $f(t)$ and pair of constants α and β.

(b) $P(D)(f(t)e^{\lambda t}) = e^{\lambda t}P(D + \lambda)(f(t))$ for every sufficiently differentiable function $f(t)$ and constant λ.

(iii) For uniqueness, list the n functions in (2) in some order $f_1, \ldots, f_n$, and consider the mapping $\phi\colon \mathbb{C}^n \to \mathbb{C}^n$ defined as follows: for each $(\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$, consider the solution

$$s_\alpha(t) = \alpha_1 f_1 + \ldots \alpha_n f_n,$$

and let

$$\phi(\alpha_1, \ldots, \alpha_n) := (s_\alpha(0), s'_\alpha(0), \ldots, s_\alpha^{(n-1)}(0)) \in \mathbb{C}^n.$$

Since taking differentiation and evaluation are both linear operations, ϕ is linear. It is surjective since we know from part (ii) that we can find a solution as a linear combination of $f_1, \ldots, f_n$ for each initial condition. Since ϕ is linear and has rank 4, i.e., $\dim(\operatorname{im}\phi) = 4$, the rank-nullity theorem says that the kernel of ϕ is trivial. So ϕ is injective. Now take any two solutions that satisfy the same initial conditions. Each of these solutions is a linear combination of the basic functions, so they have the form s_α and s_β for some $\alpha, \beta \in \mathbb{C}^n$. Since they satisfy the same initial condition, we have $\phi(\alpha) = \phi(\beta)$. Since ϕ is injective, we have $\alpha = \beta$.