Math 322 lecture for Monday, Week 5

Stability

Let $A \in M_n(\mathbb{C})$. For each eigenvalue $\lambda \in \mathbb{C}$, the *generalized eigenspace* for λ is

$$V_\lambda = \{v \in \mathbb{C}^n : (A - \lambda I)^k v = 0 \text{ for some } k > 0\}.$$

We can choose bases $\mathcal{B}_\lambda$ for the generalized eigenspaces resulting in a basis $\mathcal{B} = \cup_\lambda \mathcal{B}_\lambda$ for $\mathbb{C}^n$ with respect to which the matrix A attains its Jordan form. Define the *stable*, *center*, and *unstable* spaces for A respectively by

$$E^s = \text{Span } \cup_{\lambda:\text{Re}(\lambda)<0} \mathcal{B}_\lambda$$
$$E^c = \text{Span } \cup_{\lambda:\text{Re}(\lambda)=0} \mathcal{B}_\lambda$$
$$E^u = \text{Span } \cup_{\lambda:\text{Re}(\lambda)>0} \mathcal{B}_\lambda.$$

Since $\mathcal{B}$ is a basis, we can write

$$\mathbb{C}^n = E^s \oplus E^c \oplus E^u,$$

i.e., every $v \in \mathbb{C}^n$ can be written uniquely as $v = v_s + v_c + v_u$ where $v_s \in E^s$, $v_c \in E^c$, and $v_u \in E^u$.

If A is a real matrix and we are working over the real numbers, then define the (real) *stable*, *center*, and *unstable* spaces for A by intersecting each of E^s, E^c, and E^u with $\mathbb{R}^n$. Not that if A is real, is nonreal eigenvalues will occur in conjugate pairs $a \pm bi$, and the conjugates have the same real part. We can also adjust the basis $\mathcal{B}$ so that with respect to $\mathcal{B}$, the matrix A has its real Jordan form.

If $L : F^n \to F^n$ is a linear function and $W \subseteq F^n$, we say that W is *invariant under* L if $L(W) \subseteq W$. If M is the matrix representing L, we similarly say that W is invariant under M if $Mw \in W$ for all $w \in W$.

Proposition. Each generalized eigenspace, the stable, center, and unstable spaces are invariant under A and under e^{At} for all $t \in \mathbb{R}$.

Proof. Staring at the Jordan form for A and its exponential makes this result obvious, but we will give a formal proof. First consider the action of A. Fix an eigenvalue λ for A and consider the corresponding generalized eigenspace V_λ. Let $v \in V_\lambda$. To show that $Av \in V_\lambda$, we first let $w = (A - \lambda I)v$. We claim that $w \in V_\lambda$. To see this, take $k > 0$ such that $(A - \lambda I)^k v = 0$. Then $(A - \lambda I)^{k-1} w = 0$ (in the special case where $k = 1$, we have $(A - \lambda I)^0 w = (A - \lambda I)v = w = 0 \in V_\lambda$). Since $v, w \in V_\lambda$ and V_λ is a subspace,

$$Av = \lambda v + w \in V_\lambda.$$

This shows that V_λ is invariant under A. Now since each of the stable, center, and unstable spaces is formed by taking the linear span of bases for certain generalized eigenspaces, it follows that each of these is invariant under A. It follows that they are invariant under e^{At} by homework. $\square$

Thus, let $x(t)$ be the solution to the initial value problem $x' = Ax$, $x(0) = x_0$, i.e., let $x(t) = e^{At}x_0$. It follows that if $x_0 \in E^s$, then $x(t) \in E^s$ for all t. The solution never leaves the stable space. Similarly, a solution starting in the center or the unstable space never leaves that space. Further, from the Jordan form, one sees that

$$\begin{aligned} x_0 \in E^s \setminus \{0\} &\Longrightarrow \lim_{t\to\infty} x(t) = 0 \quad \text{and} \quad \lim_{t\to-\infty} |x(t)| = \infty \\ x_0 \in E^u \setminus \{0\} &\Longrightarrow \lim_{t\to\infty} |x(t)| = \infty \quad \text{and} \quad \lim_{t\to-\infty} x(t) = 0. \end{aligned}$$

In particular, if all eigenvalues of A have negative real part, then all solutions, no matter what the initial condition, are drawn into the origin. If all eigenvalues have positive real part, all solutions with non-zero initial condition will eventually leave any fixed compact set.

LINEAR SYSTEMS IN $\mathbb{R}^3$

Linear systems in $\mathbb{R}^3$. Let $A \in M_3(\mathbb{R})$. Then A either has three real eigenvalues (counting multiplicities) or it has a single real eigenvalue and pair of conjugate nonreal eigenvalues. Therefore, the possibilities for the Jordan form and for the solutions to $x' = Ax$ up to a linear change of coordinates are:

I. $u, v, w \in \mathbb{R}$:

$$J = \begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & w \end{pmatrix} \qquad x(t) = e^{Jt}x_0 = \begin{pmatrix} e^{ut} & 0 & 0 \\ 0 & e^{vt} & 0 \\ 0 & 0 & e^{wt} \end{pmatrix} x_0.$$

The behavior of the various trajectories will depend on the signs of u, v, w, with saddle-like behavior if they don't all have the same sign.

II. $u, v \in \mathbb{R}$:

$$J = \begin{pmatrix} u & 1 & 0 \\ 0 & u & 0 \\ 0 & 0 & v \end{pmatrix} \qquad x(t) = e^{Jt}x_0 = \begin{pmatrix} e^{ut} & te^{ut} & 0 \\ 0 & e^{ut} & 0 \\ 0 & 0 & e^{vt} \end{pmatrix} x_0.$$

III. $u \in \mathbb{R}$:

$$J = \begin{pmatrix} u & 1 & 0 \\ 0 & u & 1 \\ 0 & 0 & u \end{pmatrix} \qquad x(t) = e^{Jt}x_0 = \begin{pmatrix} e^{ut} & te^{ut} & \frac{t^2}{2}e^{ut} \\ 0 & e^{ut} & te^{ut} \\ 0 & 0 & e^{ut} \end{pmatrix} x_0.$$

IV. $a, b, u \in \mathbb{R}$ and $b \neq 0$:

$$J = \begin{pmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & u \end{pmatrix} \qquad x(t) = e^{Jt}x_0 = \begin{pmatrix} e^{at}\cos(bt) & -e^{at}\sin(bt) & 0 \\ e^{at}\sin(bt) & e^{at}\cos(bt) & 0 \\ 0 & 0 & e^{ut} \end{pmatrix} x_0.$$

An interesting special case is where $a = 0$.

We will take a look at examples of all of these in class.

NONHOMOGENEOUS SYSTEMS

Proposition. Let $A \in M_n(F)$ and consider the system

$$x'(t) = Ax(t) + b(t)$$

where $t \mapsto b(t) \in F^n$ is continuous. The solution with initial condition x_0 is

$$x(t) = e^{At}x_0 + e^{At}\int_{s=0}^{t} e^{-As}b(s)\,ds.$$

The solution is unique.

Proof. Defining $x(t)$ as above, use the product rule and the fundamental theorem of calculus to see

$$\begin{aligned} x'(t) &= \left(e^{At}x_0\right)' + \left(e^{At}\right)'\int_{s=0}^{t} e^{-As}b(s)\,ds + e^{At}\left(\int_{s=0}^{t} e^{-As}b(s)\,ds\right)' \\ &= Ae^{At}x_0 + Ae^{At}\int_{s=0}^{t} e^{-As}b(s)\,ds + e^{At}e^{-At}b(t) \\ &= Ae^{At}x_0 + Ae^{At}\int_{s=0}^{t} e^{-As}b(s)\,ds + e^{At}e^{-At}b(t) \\ &= A\left(e^{At}x_0 + e^{At}\int_{s=0}^{t} e^{-As}b(s)\,ds\right) + b(t) \\ &= Ax(t) + b(t). \end{aligned}$$

Uniqueness of the solution will be a homework problem. $\square$