Math 322 lecture for Monday, Week 2

III. A. First-order linear.

A first-order linear equation has the form

$$\frac{dy}{dt} + p(t)y = q(t).$$

It is solved using the integrating factor $e^{\int p(t)\,dt}$: multiplying the equation through by this factor gives

$$e^{\int p(t)\,dt)}\left(\frac{dy}{dt} + p(t)y\right) = e^{\int p(t)\,dt}q(t). \tag{1}$$

By the chain rule and the fundamental theorem of calculus, the left-hand side of this equation is

$$\frac{d}{dt}\left(e^{\int p(t)\,dt}y\right).$$

So we can integrate equation (1) to get

$$e^{\int p(t)\,dt}y = \int e^{\int p(t)\,dt}q(t)\,dt,$$

and then solve for y.

Example. Consider the following equation

$$\cos(t)\,y' + y = \sin(t)$$

with initial condition $y(0) = 1$. Dividing by $\cos(t)$ puts the equation into standard form (note that $\cos(t) \neq 0$ near $t = 0$):

$$y' + \sec(t)\,y = \tan(t).$$

The integrating factor is

$$e^{\int \sec(t)\,dt} = e^{\ln(\sec(t)+\tan(t))} = \sec(t) + \tan(t).$$

(Near $t = 0$, we have $\sec(t) + \tan(t) > 0$. Multiplying the equation through by the integrating factor gives

$$(\sec(t) + \tan(t))\,y' + (\sec^2(t) + \sec(t)\tan(t))\,y = (\sec(t) + \tan(t))\tan(t).$$

Integrate both sides:

$$
\begin{aligned}
(\sec(t)+\tan(t))y &= \int (\sec(t)+\tan(t))\tan(t)\,dt \\
&= \int (\sec(t)\tan(t)+\tan^2(t))\,dt \\
&= \int \sec(t)\tan(t)\,dt + \int \tan^2(t))\,dt \\
&= \sec(t) + \int \tan^2(t))\,dt \\
&= \sec(t) + \int \left(\sec^2(t)-1\right)\,dt \\
&= \sec(t)+\tan(t)-t+c
\end{aligned}
$$

Therefore,

$$y = \frac{\sec(t)+\tan(t)-t+c}{\sec(t)+\tan(t)}.$$

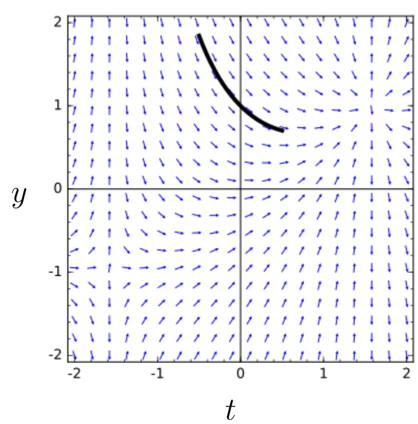
Let's write this in terms of the initial condition:

$$1 = y(0) = \frac{\sec(0)+\tan(0)+c}{\sec(0)+\tan(0)} = 1+c.$$

So $c=0$, and the solution is

$$y = \frac{\sec(t)+\tan(t)-t}{\sec(t)+\tan(t)} = 1 - \frac{t}{\sec(t)+\tan(t)}.$$

Here is a plot of the slope field and our solution:



Note the weirdness around $t = \pi/2$, where $\cos(t)$ is 0. It's exactly where we divide by zero in our calculations where the interesting stuff happens.

III. B. Bernoulli-type first-order linear.

We are now interested in solving an equation of the form

$$\frac{dy}{dt} + p(t)y = q(t)y^m,$$

where $m \neq 1$. The trick here is to reduce the equation to a standard first-order linear equation with the substitution $u = y^{1-m}$. In that case, we have

$$u' = (1-m)y^{-m}y'.$$

Multiply the original equation through by $(1-m)y^{-m}$

$$(1-m)y^{-m}y' + (1-m)p(t)y^{1-m} = (1-m)q(t)$$

and substitute:

$$u' + (1-m)p(t)u = (1-m)q(t).$$

Example. Consider the equation

$$y' = \frac{2y}{t} - t^2y^2$$

with initial condition $y(1) = -2$. This is Bernoulli-type with $m = 2$, so we make the substitution $u = y^{-1}$. This transforms the equation into the first-order linear equation

$$u' + \frac{2u}{t} = t^2.$$

The integrating factor is

$$e^{\int (2/t)\,dt} = t^2.$$

Multiply through by it and integrate:

$$\begin{aligned} t^2u' + 2tu = t^4 \quad &\Rightarrow \quad \frac{d}{dt}(t^2u) = t^4 \\ &\Rightarrow \quad t^2u = \int t^4\,dt = \frac{1}{5}t^5 + c \\ &\Rightarrow \quad \frac{t^2}{y} = \frac{1}{5}t^5 + c \end{aligned}$$

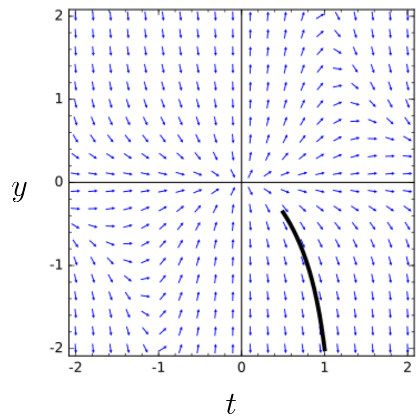
The initial condition gives us:

$$-\frac{1}{2} = \frac{1}{5} + c \quad \Rightarrow \quad c = -\frac{7}{10}.$$

The solution is

$$\frac{t^2}{y} = \frac{1}{5}t^5 - \frac{7}{10} \quad \Rightarrow \quad y = \frac{10t^2}{2t^5 - 7}.$$

The slope field and our solution:



IV. A. Linear homogeneous constant coefficients (LHCC).

We are now interested in solving a differential equation of the form

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

where $y^{(i)}$ denotes the i-th derivative of y with respect to t. The a_i are constants. The word "homogeneous" refers to the fact that a 0 appears to the right of the equals sign. Letting $D := d/dt$, we can write the above equation as

$$(D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0)y = 0$$

or just

$$P(D)y = 0$$

where P is the polynomial $P(x) = \sum_{i=0}^{n} a_i x^i$.

Main theory.

1. The solution space is linear: suppose that y_1 and y_2 are solutions, i.e., $P(D)y_1 = P(D)y_2 = 0$. Let α be a constant. Then

$$P(D)(y_1 + \alpha y_2) = P(D)y_1 + \alpha\, P(D)y_2 = 0$$

   by linearity of differentiation.

2. The "basic" solutions have the form e^{rt} (more on this later).

3. When determining which values for r are suitable, something nice happens:

$$P(D)e^{rt} = \sum_{i=0}^{n} a_i D^i e^{rt} = \sum_{i=0}^{n} a_i r^i e^{rt} = P(r)e^{rt}.$$

   Since $e^{rt} > 0$, we get a solution $P(D)e^{rt} = 0$ if and only if $P(r) = 0$. So the values for r that give solutions are exactly the zeros of the polynomial P. The polynomial $P(r)$ is called *characteristic polynomial* for the equation.

4. For uniqueness, we specify $y(t_0), \ldots, y^{(n-1)}(t_0)$.

Example. Solve

$$y'' - y' - 6y = 0$$

with initial conditions $y(0) = 0$ and $y'(0) = 1$.

SOLUTION: Find the zeros of the characteristic polynomial:

$$P(r) = r^2 - r - 6 = (r+2)(r-3) = 0 \quad \Leftrightarrow \quad r = -2, 3.$$

The general solution is

$$y = ae^{-2t} + be^{3t}.$$

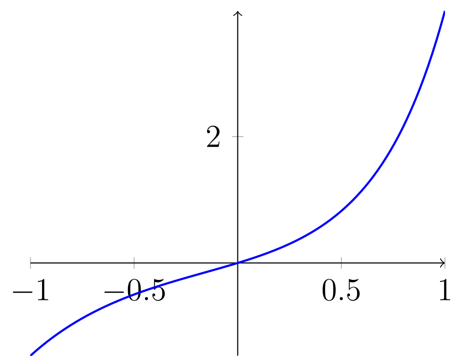
To satisfy the initial conditions, we need

$$\begin{aligned} a + b &= 0 \\ -2a + 3b &= 1. \end{aligned}$$

Solving this system gives $a = -1/5$ and $b = 1/5$. So the solution is

$$y = -\frac{1}{5}\, e^{-2t} + \frac{1}{5}\, e^{3t}.$$

A graph of the solution:


2
−1
−0.5
0.5
1