Math 322 lecture for Friday, Week 2

V. Method of undetermined coefficients.

We look at one more example of the method of undetermined coefficients. Consider the equation

$$y'' - 2y' + y = t\cos(3t).$$

We guess a particular solution of the form

$$y = (a_0 + a_1 t)\cos(3t) + (b_0 + b_1 t)\sin(3t).$$

Then

$$y' = (a_1 + 3b_0 + 3b_1 t)\cos(3t) + (-3a_0 + b_1 - 3a_1 t)\sin(3t)$$

$$y'' = (-9a_0 + 6b_1 - 9a_1 t)\cos(3t) + (-6a_1 - 9b_0 - 9b_1 t)\sin(3t)$$

So we have

$$\begin{aligned} y'' - 2y' + y = (&-8a_0 - 2a_1 - 6b_0 + 6b_1 - (8a_1 + 6b_1)t)\cos(3t) \\ &+ (6a_0 - 6a_1 - 8b_0 - 2b_1 + (6a_1 - 8b_1)t)\sin(3t) \end{aligned}$$

Set this equal to $t\cos(3t)$ and compare coefficients to get the system on linear equations

$$\begin{aligned} 0 &= -8a_0 - 2a_1 - 6b_0 + 6b_1 \\ 1 &= -8a_1 - 6b_1 \\ 0 &= 6a_0 - 6a_1 - 8b_0 - 2b_1 \\ 0 &= 6a_1 - 8b_1 \end{aligned}$$

Solving this system gives the particular solution

$$y_p = -\frac{1}{250}\,(13 + 20\,t)\cos(3\,t) - \frac{3}{250}\,(-3 + 5\,t)\sin(3\,t)\,.$$

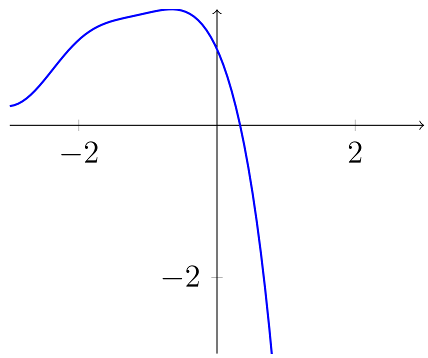
The corresponding homogeneous equation, $y'' - 2y' + y = 0$, has a general solution $ae^t + bte^t$. So the general solution to our inhomogeneous equation is

$$y = ae^t + bte^t - \frac{1}{250}\,(13 + 20\,t)\cos(3\,t) - \frac{3}{250}\,(-3 + 5\,t)\sin(3\,t)$$

Let's again consider the initial conditions $y(0) = 1$ and $y'(0) = -2$. Plugging these into the general solution and its derivative allow us to determine a and b. The result is

$$y = \frac{263}{250}\,e^t - \frac{77}{25}\,te^t - \frac{1}{250}\,(13 + 20\,t)\cos(3\,t) - \frac{3}{250}\,(-3 + 5\,t)\sin(3\,t)\,.$$

Graph of solution:



VI. A. Second-order. Given a second-order equation of the form

$$H(t, y', y'') = 0$$

i.e., missing a y-term, we can reduce the order of the equation with the substitution $v = y'$.

Example. Consider the equation

$$ty'' + 4y' = t^2.$$

Substitute $v = y'$ to get the equation

$$tv' + 4v = t^2.$$

If $t \neq 0$, this becomes the standard first-order equation

$$v' + \frac{4}{t}v = t.$$

Say $t > 0$. Then the integrating factor is $\exp\left(\int \frac{4}{t}\,dt\right) = t^4$. Multiplying through (and using the product rule), we have

$$t^4\,v' + 4t^3\,v = (t^4 v)' = t^5.$$

Integrate:

$$t^4 v = \frac{1}{6}t^6 + c.$$

Now substitute back $v = y'$:

$$t^4 y' = \frac{1}{6}t^6 + c.$$

This is separable:

$$y' = \frac{1}{6}t^2 + \frac{c}{t^4} \quad \Rightarrow \quad y = \frac{1}{18}t^3 - \frac{1}{3}\cdot\frac{c}{t^3} + b$$
$$= \frac{1}{18}t^3 + \frac{a}{t^3} + b.$$

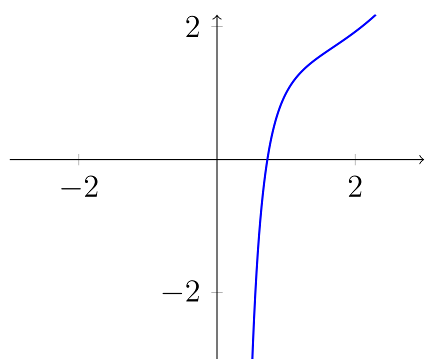
Suppose the initial conditions are $y(1) = 1$ and $y'(1) = 2$. Then

$$1 = \frac{1}{18} + a + b$$
$$2 = \frac{1}{6} - 3a,$$

which implies $a = -11/18$ and $b = 14/9$. The solution is

$$y = \frac{1}{18}t^3 - \frac{11}{18}\frac{1}{t^3} + \frac{14}{9}.$$

Graph of solution:



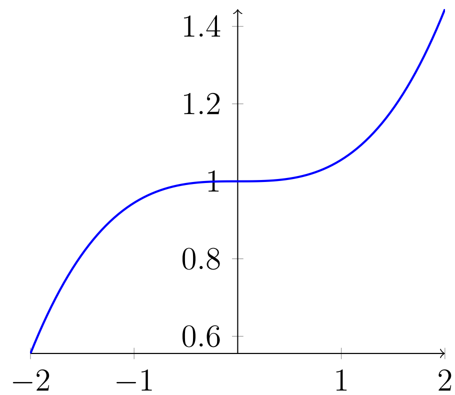
Solutions defined near $t = 0$? Our method of forcing the equation into the form of a standard first-order equation requires dividing by t, and hence, assumes that $t \neq 0$. What if we really want a solution defined near $t = 0$? My approach was to suppose the solution can be expanded in terms of a power series $y = a_0 + a_1 t + a_2 t^2 + \dots$ Plug this series into the equation $ty'' + 4y'$ and set the result equal to t^2. Now compare coefficients and hope we can solve for the a_i. If you think about it, we only need to consider series where $a_i = 0$ for $i \geq 4$. So assume $y = a_0 + a_1 t + a_2 t^2 + a_3 t^3$. We have

$$ty'' + 4y' = t(2a_2 + 6a_3 t) + 4(a_1 + 2a_2 t + 3a_3 t^2)$$
$$= 4a_1 + 10a_2 t + 18a_3 t^2.$$

Setting the result equal to t^2 and comparing coefficients gives $a_1 = a_2 = 0$, and $a_3 = 1/18$. The solution is

$$y = a_0 + \frac{1}{18}\,t^3.$$

Graph of solution with initial condition $y(0) = 1$:



Note that the only possibly initial condition for $y'(0)$ is $y'(0) = 0$ (why?). Since this is a second-order equation, we'd expect to be able to set initial conditions for both y and y'. We should try to remember to come back to this example when we talk about existence and uniqueness of solutions.

VI. B. Second-order equation.

Given a second-order equation of the form

$$H(y, y', y'') = 0$$

i.e., missing t, we again make the substitution $v = y'$, but then use the chain rule like so

$$y'' = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy}.$$

Substituting, our original equation becomes

$$H\left(y, v, v\frac{dv}{dy}\right) = 0.$$

After we find v as a function of y, we solve for y by integrating, as before.

Example. Consider the equation

$$y'' + (y')^3\, y = 0.$$

Let $v = y'$ and substitute as above to get

$$v\frac{dv}{dy} + v^3 y = 0.$$

This is first-order linear, but even better, it is separable. Supposing $v > 0$, the equation becomes

$$\frac{1}{v^2}\frac{dv}{dy} = -y.$$

Integrate:

$$\begin{aligned} \int \frac{1}{v^2}\,dv = -\int y\,dy \quad &\Rightarrow \quad -\frac{1}{v} = -\frac{1}{2}y^2 + \tilde{c} \\ &\Rightarrow \quad v = \frac{2}{y^2 - 2\tilde{c}} \\ &\Rightarrow \quad v = \frac{2}{y^2 + c}. \end{aligned}$$

Now substitute back in $v = y'$:

$$y' = \frac{2}{y^2 + c} \quad \Rightarrow \quad \int (y^2 + c)\,dy = 2\int dt \quad \Rightarrow \quad \frac{1}{3}y^3 + cy = 2t + b.$$

Suppose our initial conditions are $y(1) = 0$ and $y'(1) = 1$. Then

$$\frac{1}{2}\cdot 0^3 + c\cdot 0 = 2\cdot 1 + b \quad \Rightarrow \quad b = -2.$$

So the equation becomes

$$\frac{1}{3}y^3 + cy = -2 + 2t.$$

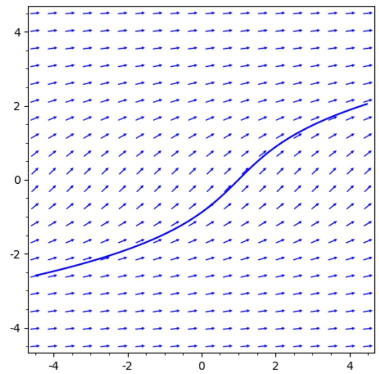
To use the second condition, take derivatives with respect to t:

$$y^2\,y' + cy' = 2.$$

Plug in $y(1) = 0$ and $y'(1) = 1$ to find $c = 2$. The solution, implicitly, is

$$\frac{1}{3}y^3 + 2y = -2 + 2t.$$

Here is a picture of the slope field and our solution:



VII. Duh.

If your method of solving a differential equation is not working due to a troublesome set of initial conditions, consider obvious/trivial solutions.

Example. We just solved the equation

$$y'' + (y')^3 y = 0.$$

for a particular set of initial conditions. If you look back at our method solution, you'll see that we can find a solution satisfying any initial conditions $y(t_0) = \alpha$ and $y'(t_0) = \beta$, except for those where $\beta = 0$. That's because we divided by $v = y'$ in the course of our solution. What do we do for the troublesome case of $\beta = 0$? Applying the "duh" method, we immediately find the solution $y = \alpha$, a constant function.

Challenge. Solve

$$y'' + (y')^3 y = t.$$

with initial condition $y(0) = 1$ and $y'(0) = 0$.