Math 322 lecture for Wednesday, Week 1

1. A. SEPARABLE EQUATIONS.

Logistic growth model. Let $P(t)$ be the size of a population at time t, and let r and K be positive constants. The logistic growth model is the differential equation

$$P'(t) = rP(t)\left(1 - \frac{P(t)}{K}\right).$$

The constant r is the *growth rate* and K is the *carrying capacity* of the population. The differential equation says the growth in population is proportional to the size of the existing population with an extra factor to account for limited resources. When the population is small (when P is much smaller then K), we see $P' \approx rP$, which we've already seen leads to exponential growth. However, as P gets close to K over time, the factor $1 - P/K$ slows the growth.

Solution. The equation is separable and can be solved using integration using the technique of partial fractions.

$$P'(t) = rP(t)\left(1 - \frac{P(t)}{K}\right) \quad \Rightarrow \quad \frac{P'(t)}{P(t)\left(1 - \frac{P(t)}{K}\right)} = r.$$

The technique of partial fractions requires us to find constants A and B such that

$$\frac{1}{P(t)\left(1 - \frac{P(t)}{K}\right)} = \frac{A}{P(t)} + \frac{B}{1 - \frac{P(t)}{K}}. \tag{1}$$

We have

$$\frac{A}{P(t)} + \frac{B(t)}{1 - \frac{P(t)}{K}} = \frac{A\left(1 - \frac{P(t)}{K}\right) + BP(t)}{P(t)\left(1 - \frac{P(t)}{K}\right)}. \tag{2}$$

Comparing numerators in equations (1) and (2), we need to adjust A and B so that

$$1 = A\left(1 - \frac{P(t)}{K}\right) + BP(t).$$

Or, rearranging:

$$1 = A + \left(-\frac{A}{K} + B\right)P(t).$$

We get an equality if

$$A = 1 \quad \text{and} \quad -\frac{A}{K} + B = 0.$$

So $A = 1$ and $B = 1/K$. Therefore, we can write (double-check!):

$$\frac{1}{P(t)\left(1 - \frac{P(t)}{K}\right)} = \frac{1}{P(t)} + \frac{1/K}{1 - \frac{P(t)}{K}}. \tag{3}$$

Back to solving the differential equation:

$$\frac{P'(t)}{P(t)\left(1 - \frac{P(t)}{K}\right)} = r \quad \Rightarrow \quad \int \frac{dP}{P\left(1 - \frac{P}{K}\right)} = \int r\, dt$$

$$\Rightarrow \quad \int \frac{dP}{P(t)\left(1 - \frac{P}{K}\right)} = rt + \text{constant}.$$

For the left-hand side, use equation (3):

$$\begin{aligned}
\int \frac{dP}{P(t)\left(1 - \frac{P}{K}\right)}\, dt &= \int \left(\frac{1}{P(t)} + \frac{1/K}{1 - \frac{P(t)}{K}}\right) dP \\
&= \int \frac{dP}{P(t)} + \frac{1}{K} \int \frac{dP}{1 - \frac{P(t)}{K}} \\
&= \ln P(t) - \ln\left(1 - \frac{P(t)}{K}\right) + \text{constant}.
\end{aligned}$$

Here, we have assumed that $0 < P < K$ (how?). Exponentiate both sides to get

$$P(t)\left(1 - \frac{P(t)}{K}\right)^{-1} = ae^{rt}$$

for some positive constant a. We now need to solve this equation for $P(t)$:

$$\begin{aligned}
ae^{rt} = P(t)\left(1 - \frac{P(t)}{K}\right)^{-1} &= \frac{KP(t)}{K - P(t)} \\
\Rightarrow ae^{rt}(K - P(t)) &= KP(t) \\
\Rightarrow aKe^{rt} &= ae^{rt}P(t) + KP(t) = (ae^{rt} + K)P(t) \\
\Rightarrow P(t) &= \frac{aKe^{rt}}{ae^{rt} + K}
\end{aligned}$$

$$\Rightarrow P(t) = \frac{aK}{a + Ke^{-rt}}.$$

We would like to express the arbitrary constant a in terms of the initial population:

$$P(0) = \frac{aKe^0}{ae^0 + K} = \frac{aK}{a + K}$$

$$\Rightarrow P(0)(a + K) = aK$$

$$\Rightarrow P(0)K = aK - P(0)a = a(K - P(0))$$

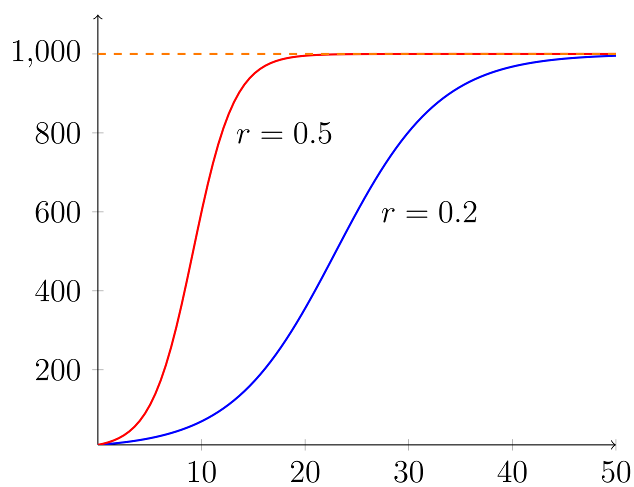
$$a = \frac{P(0)K}{K - P(0)}.$$

Substituting this expression for a and simplifying gives the final form for the solution

$$\boxed{P(t) = \frac{P(0)K}{P(0) + (K - P(0))e^{-rt}}.}$$

(**Exercise:** How would things change if $P > K$?) It's easy to see from this equation that the limiting population is

$$\lim_{t\to\infty} P(t) = K.$$



Graph of $P(t)$ with $K = 1000$ and $P(0) = 10$ and two different growth rates: $r = 0.5$ in red and $r = 0.2$ in blue.

Exercise. A state game commission releases 40 elk into a game refuge. After 5 years, the elk population is 104. The commission believes that the refuge can support no more than 4000 elk. Use a logistic model to predict the elk population in 15 years.

SOLUTION: The carrying capacity is $K = 4000$, so the logistic model in this situation is

$$P'(t) = rP(t)\left(1 - \frac{P(t)}{4000}\right)$$

where we can determine r from the additional information we're given. The initial population size is $P(0) = 40$. From the solution to the logistic equation we derived above, we have

$$\begin{aligned} P(t) &= \frac{4000P(0)}{P(0) + (4000 - P(0))e^{-rt}} \\ &= \frac{160000}{40 + 3960e^{-rt}} \\ &= \frac{4000}{1 + 99e^{-rt}} \end{aligned}$$

We are given that $P(5) = 104$. Therefore,

$$104 = P(5) = \frac{4000}{1 + 99e^{-5r}}.$$

Solve for r:

$$\begin{aligned} 104 = \frac{4000}{1 + 99e^{-5r}} \quad &\Rightarrow \quad 104(1 + 99e^{-5r}) = 4000 \\ &\Rightarrow \quad e^{-5r} = \frac{1}{99}\left(\frac{4000}{104} - 1\right) = \frac{487}{1287} \\ &\Rightarrow \quad -5r = \ln\left(\frac{487}{1287}\right) \\ &\Rightarrow \quad r \approx 0.194. \end{aligned}$$

So our model for this population is

$$P(t) = \frac{4000}{1 + 99e^{-0.194\,t}}$$

So we would predict the population after 15 years to be

$$P(15) = \frac{4000}{1 + 99e^{-0.194 \cdot 15}} \approx 626.$$

1. B. Separable—homogeneity trick.

An equation of the form

$$y' = F\left(\frac{y}{t}\right)$$

can be turned into a separable equation using the following substitution: let $v = y/t$. It follows that $y = vt$, and thus, $y' = v + tv'$ by the product rule. Then,

$$\begin{aligned} y' = F\left(\frac{y}{t}\right) \quad &\Rightarrow \quad v + tv' = F(v) \\ &\Rightarrow \quad \frac{v'}{F(v) - v} = \frac{1}{t} \\ &\Rightarrow \quad \int \frac{dv}{F(v) - v} = \int \frac{dt}{t}. \end{aligned}$$

Example. Solve

$$y' = \frac{y^2 + 2yt}{t^2}.$$

Notice that in the fraction on the right, the degree of every term in the numerator and denominator is 2. That's a sign of homogeneity. In fact, we have

$$\frac{y^2 + 2yt}{t^2} = \frac{y^2}{t^2} + \frac{2yt}{t^2} = \left(\frac{y}{t}\right)^2 + 2\left(\frac{y}{t}\right).$$

Substitute $v = y/t$ and $y' = v + tv'$ to transform the original equation into:

$$v + tv' = v^2 + 2v.$$

Separate variables and integrate. For convenience, we will assume that $y > 0$ and $t > 0$, and hence $v > 0$. Other cases can be handled similarly:

$$\frac{v'}{v^2 + v} = \frac{1}{t} \quad \Rightarrow \quad \int \frac{dv}{v^2 + v} = \int \frac{dt}{t}.$$

To integrate the left-hand side, use partial fractions:

$$\int \frac{dv}{v^2 + v} = \int \frac{dv}{v(v+1)} = \int \left(\frac{1}{v} - \frac{1}{v+1}\right) dv$$

$$= \ln(v) - \ln(v+1) + \tilde{c}$$
$$= \ln\left(\frac{v}{v+1}\right) + \tilde{c}.$$

We have found that
$$\ln\left(\frac{v}{v+1}\right) = \ln(t) + c.$$

Exponentiate and solve for v:
$$\frac{v}{v+1} = at. \quad \Rightarrow \quad v = \frac{at}{1-at}.$$

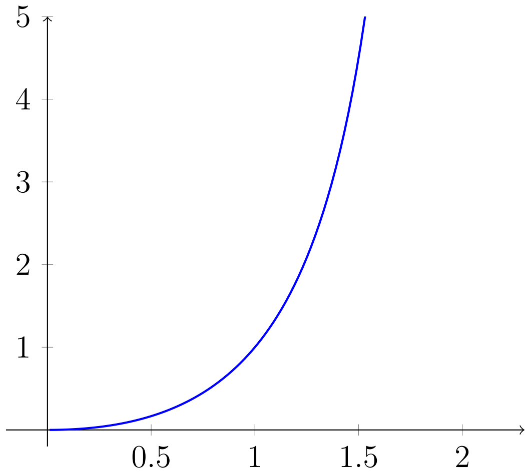
Since $v = y/t$, we get
$$y = \frac{at^2}{1-at}.$$

Considering an initial condition at $t = 0$ doesn't make much sense (why?). Let's write our solution in terms of an initial condition $I = y(1)$:
$$I = y(1) = \frac{a}{1-a} \quad \Rightarrow \quad a = \frac{I}{1+I}.$$

Substituting gives
$$y = \frac{I\,t^2}{I+1-It}.$$

For instance, if $I = 1$, we get the solution
$$y = \frac{t^2}{2-t},$$

which is defined on the open interval $(-\infty, 2)$, however, recall that at some point along the way, we assumed $t > 0$. And, in fact, our original equation is undefined at $t = 0$. So the appropriate interval for this solution is $(0, 2)$:



Solution to $y' = (y^2 + yt)/t^2$ with $y(1) = 1$.

Here is the Sage code for solving this equation:

```
sage: t = var('t')
sage: y = function('y')(t)
sage: desolve(diff(y,t)-(y^2+2*y*t)/t^2,y)
-(t^2 + t*y(t))/y(t) &=  _C
sage: desolve(diff(y,t)-(y^2+2*y*t)/t^2,y,ics=[1,1])
-(t^2 + t*y(t))/y(t) &=  -2
```

For the second call to desolve, I've included initial conditions, y first and t second:

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ics = [y(t_0), t_0].
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