Math 322 lecture for Monday, Week 1

Homepage. Please see our course homepage. In particular, please follow the "Course information" link and carefully read the material there.

Text. Our text is *Differential equations and dynamical systems*, edition 3, by Lawrence Perko.

Overview. This is a course in differential equations for advanced undergraduates. Here is a rough outline of what we'll cover:

1. **Elementary methods.** We will spend two weeks to cover the major part of what one might do in a first course in differential equations in a lower-level course.

2. **Linear theory.** We will then learn how to solve systems of linear differential equations with constant coefficients. The key is exponentiation of matrices and the Jordan form.

3. **Local nonlinear theory.** Most nonlinear systems cannot be solved. So we are interested in describing the qualitative behavior of such systems. The main idea is to approximate nonlinear systems with linear systems. We will cover the important existence-uniqueness theorems.

4. **Global theory.** At the end of the course, we will study limits of trajectories and topological properties for systems of ODEs.

First goal: elementary methods. For this part of the course, we will not have a text. The main source for information will be the lecture notes and handouts. There is also plenty of material readily available online. My hope is that this will be a time to have fun doing lots of computations by hand. We will cover the six methods from the handout First recipes, starting with separable equations. In the following, we will generally think of y as a real-valued function of t.

I. Separable equations. A separable differential equation can has the form (or can be manipulated to have the form)

$$p(y)\frac{dy}{dt} = q(t).$$

It is solved by integration:

$$\int p(y)\,dy = \int q(t)\,dt.$$

EXAMPLES. Consider the differential equation

$$y' = \frac{3t}{y}.$$

It's separable since we can get the ys on one side of the equality and the ts on the other:

$$yy' = 3t.$$

Integrate:

$$\int y(t)y'(t)\,dt = \int 3t\,dt.$$

Forgetting about constant until the end, the right-hand side is

$$\int 3t\,dt = \frac{3}{2}t^2$$

For the left-hand side, make the substitution $u = y(t)$. So $du = y'(t)\,dt$. Substituting gives:

$$\int y(t)y'(t)\,dt = \int u\,du = \frac{1}{2}u^2 = \frac{1}{2}y^2.$$

Setting the two sides equal and adding a constant gives the most general solution:

$$\frac{1}{2}y^2 = \frac{3}{2}t^2 + \tilde{c}$$

or, equivalently,

$$\boxed{y(t)^2 = 3t^2 + c}$$

for some constant c.

(An alternative way to integrate:

$$\int y\,dy = \int 3t\,dt \quad \Rightarrow \quad \frac{1}{2}y^2 = \frac{3}{2}t^2 + c.)$$

To find a particular solution, we can impose an initial condition. For instance, if $y(0) = 5$, then

$$25 = y(0)^2 = 3 \cdot 0^2 + c \quad \Rightarrow \quad c = 25,$$

and the solution is defined implicitly by

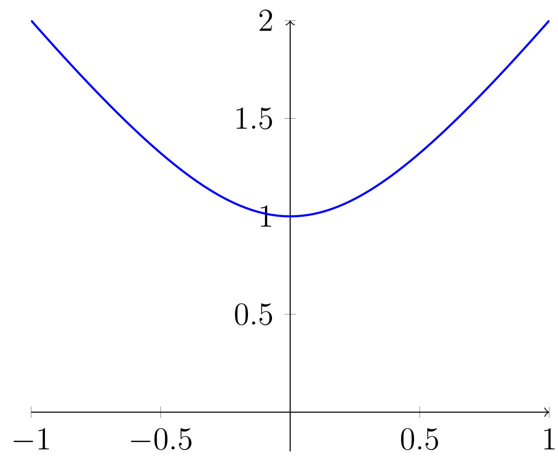
$$y(t)^2 = 3t^2 + 25.$$

Thus, $y(t) = \pm\sqrt{3t^2+25}$. Since we want $y(0) = 5$, we must choose the positive solution:

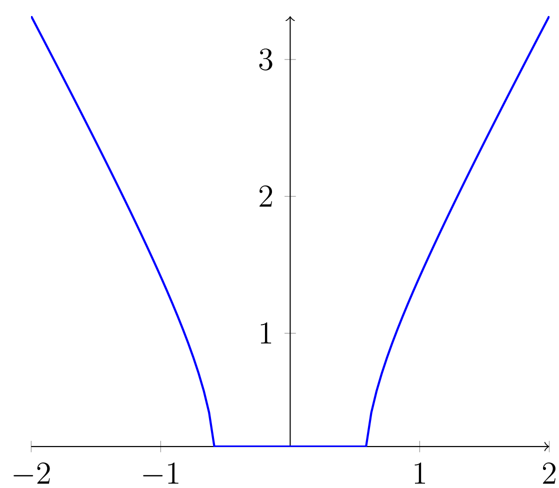
$$y(t) = \sqrt{3t^2+25}.$$

It is a solution for all $t \in \mathbb{R}$. If you initial condition were $y(0) = -5$, the solution would be $y(t) = -\sqrt{3t^2+25}$, again for all $t \in \mathbb{R}$.

There are, qualitatively, two types of behavior for solutions of this differential equation depending on whether c is positive or negative.



Graph of $y(t) = \sqrt{3t^2+1}$.



Graph of $y(t) = \sqrt{3t^2-1}$.

For an example where $c < 0$, suppose the initial condition is $y(-1) = \sqrt{2}$. Then

$$2 = y(-1)^2 = 3 \cdot 1^2 + c \quad \Rightarrow \quad c = -1,$$

and the implicit solution is

$$y(t)^2 = 3t^2 - 1.$$

The solution (with the given initial condition) is

$$y(t) = \sqrt{3t^2 - 1},$$

which makes sense for $3t^2 \geq 1$, i.e., $t \geq \sqrt{3}/3$ and $t \leq -\sqrt{3}/3$. Since our initial condition is at $t = -1$, the maximal interval for the solution is $(-\infty, -\sqrt{3}/3)$.

Exponential growth and decay model. Let $y(t)$ now denote the size of a population, varying over time. What happens if we assume that the rate of growth of the population is proportional to the size of the population? The rate of growth of the population is $y'(t)$ and the size of the population is $y(t)$. To say they are proportional is to say there is a constant r such that

$$y'(t) = ry(t).$$

This is a separable equation, which is easy to solve:

$$y'(t) = ry(t) \quad \Rightarrow \quad \frac{y'(t)}{y(t)} = r \quad \Rightarrow \quad \int \frac{y'(t)}{y(t)}\,dt = \int r\,dt.$$

Integrate, then solve for y:

$$\ln|y(t)| = rt + c \quad \Rightarrow \quad |y(t)| = e^{rt+c} = e^c e^{rt} = ae^{rt},$$

where a a positive constant. So the solution is

$$y(t) = \begin{cases} ae^{rt} & \text{if } y > 0 \\ -ae^{rt} & \text{if } y < 0. \end{cases}$$

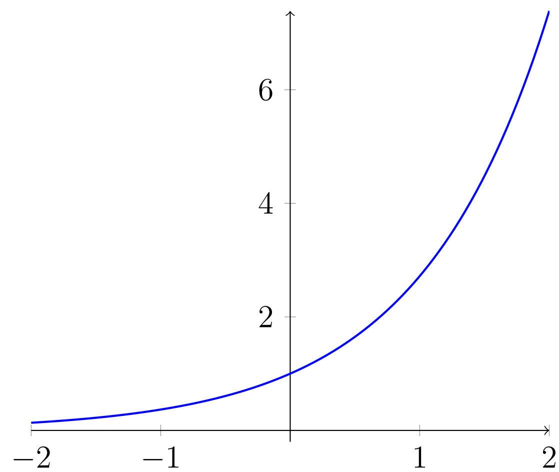
where a is positive. But we can combine these two solutions into the single solution $y(t) = ae^{rt}$ by letting a be any nonzero real number. Setting $t = 0$, we see

$$y(0) = ae^0 = a.$$

Hence, a is the initial population. So we might write the solution as

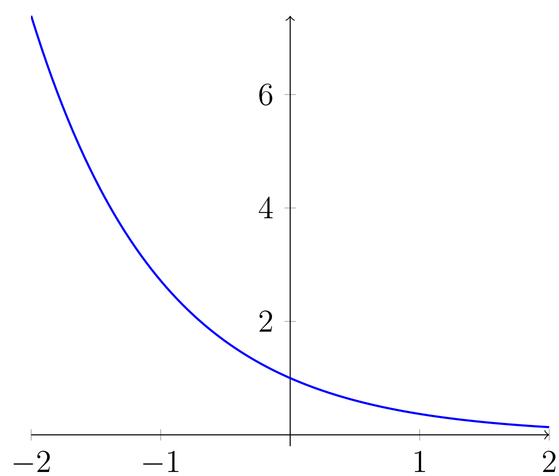
$$y(t) = y_0 e^{rt}.$$

For instance, if $y_0 = r = 1$, we get the picture below:



Graph of $y(t) = e^t$.

If $y_0 = 1$ and $r = -1$, we get:



Graph of $y(t) = e^t$.

In performing the integration, *we assumed that* y *was never zero in the range over which we integrated*. What if the initial condition is $y(t_0) = 0$ for some time t_0? One solution then is to take $y(t) = 0$ for all t. Again, the equation $y(t) = y_0 e^{rt}$ works. Is this the only solution? We'll focus on this question later in the course.

Example. If $y(t) = ae^{rt}$ with $y(0) = a \neq 0$ at what time t has the population doubled?

SOLUTION: The initial population size is a. So we are trying to find the time t when $y(t) = 2a$, so we need to solve

$$ae^{rt} = 2a.$$

Since $a \neq 0$, we need to solve

$$e^{rt} = 2$$

for t. Taking logs,

$$\ln(2) = \ln(e^{rt}) = rt.$$

Hence, assuming $r \neq 0$,

$$t = \frac{\ln(2)}{r}.$$

If $r = 0$, then $y(t) = a$ for all t, and the population never doubles.

Population model based on Newton's law of cooling. Suppose now that the rate of change of the population is governed by the differential equation

$$y'(t) = r(S - y(t))$$

where r and S are positive constants.

Problems:

1. When is the population increasing? Decreasing?

   ANSWER: We have

   $$y'(t) = r(S - y(t)) > 0 \quad \Leftrightarrow \quad S - y(t) > 0 \quad \Leftrightarrow \quad S > y(t).$$

   So the population is increasing whenever it's less that S and decreasing whenever it's larger than S.

2. What is the long-term behavior of the population?

   ANSWER: Given the answer to the previous problem it seems like the population should tend towards S.

3. Solve the equation assuming $y < S$.

   SOLUTION: The equation is separable:

   $$\begin{aligned}\int \frac{dy}{S-y} = \int r\,dt \quad &\Rightarrow \quad -\ln(S-y) = rt + c \\ &\Rightarrow \quad S - y = ae^{-rt} \\ &\Rightarrow \quad y = S - ae^{-rt}.\end{aligned}$$

Note that $y(t) \to S$ as $t \to \infty$.

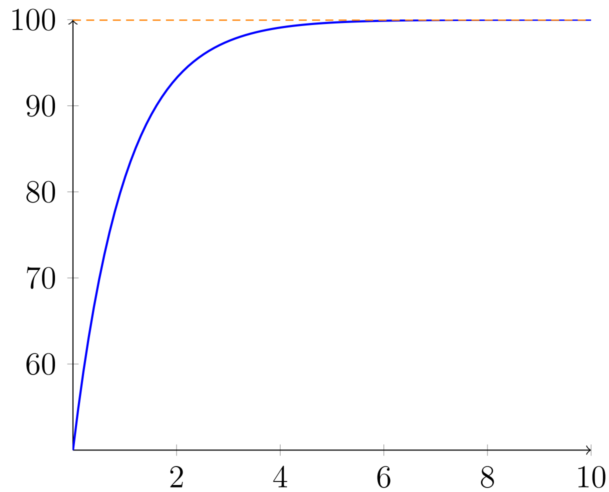
Let's now make the initial population explicit in the solution. Say I is the initial population. Then

$$I = y(0) = S - ae^0 = S - a \quad \Rightarrow \quad a = S - I.$$

Our final form for the solution is

$$\boxed{y(t) = S - (S - I)e^{-rt}},$$

where $I = y(0)$ is the initial population.



Graph of $y(t) = S - (S - I)e^{-rt}$ with $S = 100$, $I = 50$, and $r = 1$.