Math 322 Homework 12

PROBLEM 1. For this problem please refer to the notes for Friday, Week 11, and Monday, Week 12. In the first part of these notes, we described how to induce a flow on the unit sphere centered at the origin in $\mathbb{R}^3$ and then analyze a critical point $(a, b, 0)$ on the equator by projecting the flow to the $x = 1$ plane. We gave coordinates u, v to the $x = 1$ plane and derived a system of differential equations in these coordinates. To analyze the critical point $(a, b, 0)$ on the equator, we could then analyze the critical point $\left(\frac{b}{a}, 0\right)$ in the u, v-plane. Antipodal points have similar behavior except that directions may be reversed depending on the parity of $d = \max\{\deg P, \deg Q\}$, using the notation in the handout.

Carry out the same analysis for projection to the $y = 1$ plane (this time assuming $b \neq 0$) to derive the system of equations

$$u' = v^d \left(P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right) \right)$$

$$v' = -v^{d+1} Q\left(\frac{u}{v}, \frac{1}{v}\right).$$

Explain your steps.

PROBLEM 2. For each system below

i. Find and classify each critical point in the plane (sink, source, saddle, etc.)

ii. Determine and analyze the critical points at infinity (projecting to the $x = 1$ plane unless the critical point is $(0, \pm 1, 0)$, in which case, project to the $y = 1$ plane).

iii. Draw the global phase portrait. (For ease of TeX-ing, I would recommend using a tablet or hand-drawing the phase portrait and taking a photo. Then include the resulting filed using `\includegraphics`.)

iv. Use a computer to create a picture of the vector field or flow. (In Sage, you may want to use `streamline_plot` instead of `plot_vector_field`.)

(a)

$$x' = 2x$$
$$y' = y.$$

(b)

$$x' = x - y$$
$$y' = x + y.$$

(c)

$$x' = 2x - 2xy$$
$$y' = 2y - x^2 + y^2.$$

(d) In this problem, you'll get critical points at ∞ that aren't isolated. Just analyze the one at $(1, 0, 0)$.

$$x' = x$$
$$y' = y.$$