Math 322 Homework 11

Problem 1. Use the index formula

$$I_f(C) = \frac{1}{2\pi} \oint_C \frac{P\,dQ - Q\,dP}{P^2 + Q^2}$$

to compute the index of a saddle at the origin.

Problem 2. Let $z = x + iy$ and consider the vector field in the complex plane given by

$$z' = x' + iy' = z^k$$

where $k \in \mathbb{Z}$. Thus, we are interested in the system

$$\begin{aligned} x' &= \mathrm{Re}(z^k) \\ y' &= \mathrm{Im}(z^k). \end{aligned}$$

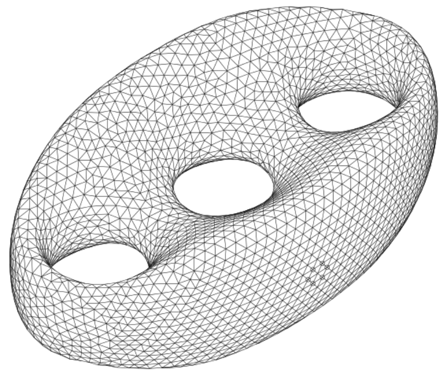
The origin is the unique critical point for the system.

(a) What is the index of the origin, in general, for the system $z' = z^k$? Explain. Hint: write $z = re^{i\theta}$ and imagine traveling counterclockwise around a unit circle centered at the origin. At the point $e^{i\theta}$ on the circle, what is the angle of the vector $z^k$? What is the total change as your go around the circle?

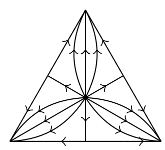
(b) Draw the vector field for the case $k = 3$.

(c) Draw the vector field for the case $k = -3$. (For a check, you could make sure for yourself that the index in these last two parts agrees with your answer to part (a).)

Problem 3. Let $M$ be a compact oriented two-dimensional manifold. It turns out that this means that $M$ is a donut with $g$ holes for some $g \in \mathbb{N}$. Triangulate $M$: draw triangles on the surface so that every point in $M$ is in some triangle and if two triangles meet, they either do so vertex-to-vertex or edge-to-edge:



Let $V$ be the number of vertices, $E$ the number of edges, and $F$ the number of faces of the triangulation. Create a vector field on each triangle vanishing at seven points, as shown below:



Consider the resulting vector field on $M$. What is the sum of the indices of the critical points in terms of $V$, $E$, and $F$? Explain.

PROBLEM 4. Let $D := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ be the unit disk in the plane centered at the origin. Suppose that $\phi\colon D \to D$ is a smooth function. We would like to use index theory to prove that $\phi$ has a fixed point on $D$, i.e., there is a point $p \in D$ such that $\phi(p) = p$.

(a) Define a vector field $f$ on $D$ by

$$f(p) := \phi(p) - p.$$

Let $C$ be the boundary of $D$, and suppose that $f$ does not vanish anywhere on $C$. What is $I_f(C)$, the index of $C$ relative to $f$? Explain.

(b) Explain why part (a) would lead to a contradiction if $\phi$ had no fixed points.