Math 322 Homework 10

PROBLEM 1. The equilibrium point at the origin for the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

is a center (since the eigenvalues of the linear function on the right are $\pm i$). In this problem, we show that perturbing the system a little can lead to various types of equilibrium points at the origin.

Establish the following results using the Liapunov function $V(x, y) = x^2 + y^2$:

(a) The origin is an asymptotically stable equilibrium point for the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -x^3 - xy^2 \\ -y^3 - x^2y \end{pmatrix}.$$

(b) The origin is an unstable equilibrium point for the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x^3 + xy^2 \\ y^3 + x^2y \end{pmatrix}.$$

(c) The origin is a stable equilibrium point which is not asymptotically stable for the system. What can you say about the solution trajectories in this case?

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -xy \\ x^2 \end{pmatrix}.$$

(d) Draw the vector field or the flow diagrams for the above three systems in order to verify your results.

PROBLEM 2. The point of this problem is to give an example of an asymptotically stable equilibrium point that is not stable. Consider the following system, defined on $\mathbb{R}^2 \setminus \{(0,0)\}$:

$$x' = x - y - x(x^2 + y^2) + \frac{xy}{\sqrt{x^2 + y^2}}$$

$$y' = x + y - y(x^2 + y^2) - \frac{x^2}{\sqrt{x^2 + y^2}}.$$

(a) Converting to polar coordinates, $x = r\cos(\theta)$, $y = r\sin(\theta)$, since $r^2 = x^2 + y^2$, we have

$$rr' = xx' + yy',$$

which makes it easy to compute $r'$ directly from the equation for the system. Similarly, it is easy to check (by substituting the polar coordinates for $x$ and $y$) that

$$r^2\theta' = xy' - x'y.$$

For this problem, find an expression for $r'$ completely in terms of $r$ and an expression for $\theta'$ completely in terms of $\theta$. You may have use for the half-angle formula

$$\sin^2(\theta/2) = \frac{1 - \cos(\theta)}{2}.$$

(b) Solve the converted system completely, i.e., for each initial condition. Your solution should write $r^2$ as a function of time and the initial condition $r_0$ and should write $\cot(\theta/2)$ as a function of time and the initial condition $\theta_0$. Show your work (you may need to review integration using partial fractions and integration of trig functions.) Don't forget the "duh" solutions where $\theta = 0$ or $r = 1$.

(c) Show that the equilibrium point $(1, 0)$ is asymptotically stable but not stable. (It might help to remember what the plot of the cotangent function looks like.)

(d) Create a phase portrait (picture of the flow of the vector field). (With Sage, you can use `streamline_plot`.)