Math 322 Homework 9

Problem 1. Consider the system

$$x' = -x$$
$$y' = y + x^3.$$

(a) Use the method of successive approximations given in class to compute the stable manifold for the equilibrium at the origin. (It converges fairly quickly.)

(b) Compute the exact solution with arbitrary initial condition (x_0, y_0) using methods from the beginning of the semester. (Any constants you use should be expressed in terms of x_0 and y_0.) Show your work.

(c) From your answer to part (b), find the initial conditions (x_0, y_0) such that corresponding solution converges to $(0, 0)$ as $t \to \infty$. (Your answer to part (a) should provide a check!)

(d) From your answer to part (b), find the initial conditions (x_0, y_0) such that corresponding solution converges to $(0, 0)$ as $t \to -\infty$, the unstable manifold.

(e) Draw the vector field with superimposed stable and unstable manifolds at $(0, 0)$.

Problem 2. Solve the system

$$x' = -x$$
$$y' = -y + x^2$$
$$z' = z + y^2$$

using methods from the beginning of the semester, and use your solution to show that the points in the stable manifold satisfy

$$z = -\frac{1}{3}y^2 - \frac{1}{6}x^2 y - \frac{1}{30}x^4$$

and the points on the unstable manifold satisfy

$$x = y = 0.$$

(It would be great if anyone could come up with a nice visualization of this!)

Problem 3. **Sinks, sources, and saddles.** Consider the differential equation $x' = f(x)$. Any point x_0 in the domain of f such that $f(x_0) = 0$ is called an *equilibrium point* or *critical point* for the equation. An equilibrium point x_0 is called a *sink* if all eigenvalues of the Jacobian matrix $Jf(x_0)$ have negative real part; it is called a *source* if all eigenvalues of $Jf(x_0)$ have positive real part; and it is called a *saddle* if $Jf(x_0)$ has at least one eigenvalue with positive real part, at least one eigenvalue with negative real part, and no eigenvalues with zero real part. Equilibrium points like these, with no eigenvalue having real part equal to zero, are called *hyperbolic equilibrium points*. The Hartman-Grobman theorem says that near a hyperbolic equilibrium point x_0, a nonlinear system $x' = f(x)$ will behave qualitatively like the associated linear system $x' = Jf(x_0)x$ does near the origin.

For the systems with $f(x)$ as follows, (i) find all equilibrium points, and (ii) classify each hyperbolic equilibrium point as a sink, source, or saddle, and (iii) draw a picture of the vector field which includes all of the equilibrium points.

(a) $\begin{pmatrix} x_1 - x_1x_2 \\ x_2 - x_1^2 \end{pmatrix}$

(b) $\begin{pmatrix} -4x_2 + 2x_1x_2 - 8 \\ 4x_2^2 - x_1^2 \end{pmatrix}$.