Math 322 Homework 8

Problem 1. We recently considered linearizing a system $x' = f(x)$ at an equilibrium point x_0, i.e., at a point where $f(x_0) = 0$ by considering the system $x' = Jf(x_0)x$ where $Jf(x_0)$ is the Jacobian matrix of f as the origin. We could hope that the linearized system would determine the nature of the equilibrium point for the original system. The following example shows that hope is unfounded if the linearized system has a center at the origin. (It turns out that in general for this case, x_0 is either a center or a focus for the original system.) Consider the system

$$\begin{aligned} x' &= -y + xy^2 \\ y' &= x + y^3 \end{aligned}$$

(a) What is the linearized system at the equilibrium point $(0,0)$?

(b) What are the eigenvalues for the linearized system? (You'll find the real parts of these are 0, and hence describe a center.)

(c) Convert to polar coordinates: $r^2 := x^2 + y^2$, $x = r\cos(\theta)$, and $y = r\sin(\theta)$. For the original system, prove that $r' = r^3\sin^2(\theta)$ and $\theta' = 1$.

(d) What does part (c) say about solutions for our original system?

(e) Use a computer to plot the vector field. Here is sample code for Sage:

```
x,y = var('a,b')
plot_vector_field((-y,x),(x,-2,2),(y,-2,2),aspect_ratio=1)

    or

streamline_plot((-y,x),(x,-2,2),(y,-2,2),aspect_ratio=1,density=2)
```

(Please see the posting for last week's homework on our class homepage for instructions on including a plot in your LaTeX document.)

Problem 2. Consider the system

$$\begin{aligned} x' &= x - xy \\ y' &= y - x^2 \end{aligned}$$

(a) Find all the equilibrium points x_0, i.e, the points x_0 at which $f(x_0) = 0$.

(b) For each equilibrium point x_0, compute the Jacobian $Jf(x_0)x$ and consider the linear system

$$\begin{aligned} x' &= Jf(x_0)x \\ x(0) &= x_0 \end{aligned}$$

Classify the origin for this linearized system as a saddle, a stable or unstable node, a stable or unstable focus, or a center.

(c) Use a computer to plot the vector field, including all the equilibrium points.

Projective space. Here is an example of an important manifold, n-dimensional projective space, $\mathbb{P}^n$. As a set, $\mathbb{P}^n$ is the collection of one-dimensional linear subspaces of $\mathbb{R}^{n+1}$, i.e., all lines through the origin in $\mathbb{R}^{n+1}$. Every one-dimensional subspace is the same thing as the span of some nonzero vector. Two nonzero vectors $x, y \in \mathbb{R}^{n+1}$ determine the same one-dimensional subspace if and only if there is a nonzero scalar λ such that $x = \lambda y$, and in this case we will write $x \sim y$. Then $\sim$ is an equivalence relation on nonzero elements of $\mathbb{R}^{n+1}$. The equivalence classes are in one-to-one correspondence with one-dimensional subspaces and hence with points in $\mathbb{P}^n$. Therefore, sometimes one will see the following definition for projective space

$$\mathbb{P}^n = \left(\mathbb{R}^{n+1} \setminus \{0\}\right) / \left(x \sim \lambda x,\ \lambda \neq 0\right).$$

Abusing notation, one usually refers to a point in $\mathbb{P}^n$ as $x = (x_0, \ldots, x_n)$ when one really means the one-dimensional space spanned by x. In that case, $x_0, \ldots, x_n$ are called the *homogeneous coordinates* of the point in projective space.

Recall that a manifold is a connected metric space M with an *atlas*. The atlas is a collection of pairs (h_α, U_α) where each U_α is an open subset of M and h_α is a homeorphism of U_α to some open subset V_α of $\mathbb{R}^n$. We require that the union of the U_α is M and if $U_\alpha \cap U_\beta \neq \emptyset$, then the *transition function*

$$h_\beta \circ h_\alpha^{-1} : h_\alpha(U_\alpha \cap U_\beta) \to h_\beta(U_\alpha \cap U_\beta)$$

is a differentiable function. The standard atlas for $\mathbb{P}^n$ consists of $n+1$ charts (h_i, U_i) where, for $i = 0, \ldots, n$,

$$U_i = \{(x_0, \ldots, x_n) \in \mathbb{P}^n : x_i \neq 0\}$$

and

$$\begin{array}{rccc} h_i\colon & U_i & \to & \mathbb{R}^n \\ & (x_0, \ldots, x_n) & \mapsto & \left(\dfrac{x_0}{x_i}, \ldots, \dfrac{\widehat{x_i}}{x_i}, \ldots, \dfrac{x_n}{x_i}\right) \end{array}$$

where the symbol $\widehat{\ }$ is used to denote omitting the i-th term. Also note that U_i is well-defined since if $x_i \neq 0$ then $\lambda x_i \neq 0$ for every $\lambda \neq 0$.

For the following exercises, we let $n = 2$ and consider the projective plane, $\mathbb{P}^2$.

(a) Explicitly describe $h_i(x_0, x_1, x_2)$ for $i = 0, 1, 2$.

(b) Compute the inverse of h_0, mapping $\mathbb{R}^2 \to U_0$.

(c) Compute the transition function $h_1 \circ h_0^{-1}$.

(d) Compute the Jacobian matrix for the transition function $h_1 \circ h_0^{-1}$, and explain why its entries are continuously differentiable for each $p \in h_0(U_0 \cap U_1)$? (Hence, this transition function is continuously differentiable, and by symmetry, so are all the others.)