Math 322 Homework 7

Problem 1. Let X, Y be subsets of normed linear spaces $(V, \| \|_V)$ and $(W, \| \|_W)$, respectively, and suppose $f\colon X \to Y$. Then f is *continuous* if for all $u \in X$ and for all $\varepsilon > 0$, there exists $\delta = \delta(u, \varepsilon) > 0$ such that $\|u - v\|_V < \delta$ implies $\|f(u) - f(v)\|_W < \varepsilon$.

(a) For any normed linear space $(V, \| \|)$, prove that $\| \|\colon V \to \mathbb{R}$ is continuous (with the usual norm on $\mathbb{R}$).

(b) Let X be a subset of a normed linear space $(V, \| \|)$. Suppose $T\colon X \to X$ is a contraction mapping. Prove that T is continuous.

Problem 2. Our existence-uniqueness theorem applies to an initial value problem

$$\begin{aligned} x' &= f(x) \\ x(0) &= x_0 \end{aligned}$$

where f is a continuously differentiable function. If f is just continuous, it no longer applies. Here is an example: consider the initial value problem

$$\begin{aligned} x' &= 2\sqrt{x} \\ x(0) &= 0. \end{aligned}$$

For each $a > 0$, define

$$x_a(t) := \begin{cases} 0 & \text{if } t \leq a \\ (t-a)^2 & \text{if } t > a. \end{cases}$$

(a) Sketch the graph of $x_a(t)$.

(b) Each $x_a(t)$ is clearly differentiable away from $t = a$. Use the definition of the derivative to prove that $x_a(t)$ is differentiable at $t = a$.

(c) Show that each $x_a(t)$ solves the initial value problem.

Problem 3. Read Theorems 1 and 2 in Section 2.4 (*The Maximal Interval of Existence*). Consider the initial value problem

$$x_1' = x_1^2 \qquad\qquad x_1(0) = 1$$

$$x_2' = x_2 + \frac{1}{x_1} \qquad\qquad x_2(0) = 1.$$

(a) Solve the initial value problem showing your technique.

(b) What is the maximal interval of existence (α, β)?

(c) Use a computer to draw the vector field and your solution in a single plot.

(d) How is Theorem 2 exemplified by your solution?

(e) What is the speed of your solution at each time t in the interval of existence?