Math 322 Homework 6

Problem 1. Let $A \in M_n(F)$ and let $t \mapsto b(t) \in F^n$ be continuous. Let $x_0 \in F^n$, and let u and v be solutions to the initial value problem

$$x'(t) = Ax(t) + b(t) \quad \text{and} \quad x(0) = x_0. \tag{1}$$

We have shown that for each initial condition $y_0 \in F^n$, that the solution to $y'(t) = Ay(t)$ with $y(0) = y_0$ is unique. Use this result to show that $u = v$ (i.e., the solution to system (1) is unique.)

Problem 2. We found that the solution to the forced harmonic oscillator problem

$$x'' = -x + f(t)$$

has the solution

$$x(t) = x(0)\cos(t) + x'(0)\sin(t) + \int_{s=0}^{t} f(s)\sin(t-s)\,ds.$$

We also saw by integrating that in the case $f(t) = \cos(\omega t)$, the solution is

$$x(0)\cos(t) + x'(0)\sin(t) + \frac{\cos(\omega t) - \cos(t)}{1-\omega^2}.$$

While solving this equation, at some point we assumed $\omega \neq \pm 1$.

(a) Go back to our solution and revise it to get a solution in the case where $\omega = 1$, and thus solve the forced harmonic oscillator problem with $f(t) = \cos(t)$. Use the identity

$$\sin(\theta + \psi) + \sin(\theta - \psi) = 2\cos(\psi)\sin(\theta),$$

and show your work.

(b) Graph the solution with initial condition $x(0) = x'(0) = 1$, enough to get a qualitative sense of the nature of the solution.

Problem 3. Consider the n-th order differential equation with constant coefficients

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

The characteristic polynomial for the equation is $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. Define the differential operator $D = \frac{d}{dt}$. Then our differential equation may be written

$$P(D)y = 0.$$

Suppose that P factors as $P(x) = \prod_{j=1}^{k}(x - \lambda_j)^{m_j}$ where the λ_j are distinct. We would like to show that the *basic functions* for our equation,

$$\left\{t^j e^{\lambda_i t} : 0 \leq j \leq m_i - 1,\ 1 \leq i \leq k\right\}$$

are solutions. So we need to show for each i that

$$P(D)(t^\ell e^{\lambda_i t}) = 0 \tag{2}$$

for $0 \leq \ell \leq m_i - 1$. We do this in steps.

(a) Prove by induction that for every sufficiently differentiable function $f(t)$, we have

$$(D - \lambda)^k (f(t)e^{\lambda t}) = e^{\lambda t} D^k f(t)$$

for $k \geq 0$.

(b) Use the above result to prove that for each i, equation (2) holds for $0 \leq \ell \leq m_i - 1$. You may use the fact that since D commutes with constants and with itself,

$$(D - \lambda)(D - \mu) = (D - \mu)(D - \lambda).$$

Problem 4. Let $f(t)$ be a real-valued integrable function on some open interval I containing 0, and let $x_0 \in \mathbb{R}$. Consider the initial value problem

$$\begin{aligned} x'(t) &= f(x(t)) \\ x(0) &= x_0. \end{aligned}$$

By the fundamental theorem of calculus,

$$x(t) := x_0 + \int_{s=0}^{t} f(x(s))\, ds$$

is a solution. (You could check by computing $x'(t)$ and $x(0)$.) Even if we cannot compute the integral directly, we can attempt to find a solution via the method of *successive approximations*. Define

$$u_0(t) := x_0$$

and for $k \geq 0$,

$$u_{k+1}(t) := x_0 + \int_{s=0}^{t} f(u_k(s)).$$

Consider the case where $f(t) = \lambda t$ for some $\lambda \in \mathbb{R}$.

(a) Apply the method of successive approximations to find u_1, u_2, and u_3.

(b) Identify $\lim_{n\to\infty} u_n$. No proof is necessary.

(c) Solve the initial value problem exactly using methods we already know. (Your solution should agree with the limit you just calculated.)