Math 322 Homework 5

**Problem 1.** Recall that for the Jordan block matrix $J_k(\lambda)$, we have

$$(J_k(\lambda) - \lambda I_k)e_1 = e_1,$$

i.e., $e_1$ is an eigenvector for $J_k(\lambda)$, and

$$(J_k(\lambda) - \lambda I_k)e_i = e_{i-1}$$

for $i = 2, \ldots, k$. So in order to put an $n \times n$ matrix $A$ into Jordan form, for each eigenvalue $\lambda$, we look for vectors $v_i$ such that

$$(A - \lambda I_n)v_1 = v_1,$$

and

$$(A - \lambda I_n)v_i = v_{i-1}$$

for $i = 2, \ldots$ These $v_i$ will end up as columns in a matrix $P$ for which $P^{-1}AP$ is in Jordan from.

This problem will consider a simple example of this procedure. Let

$$A = \begin{pmatrix} 1 & 4 \\ -1 & 5 \end{pmatrix}.$$

*Do all of the following exercises by hand, and show your work.*

(a) Compute the characteristic polynomial $p(x)$ for $A$, and factor it to find the single eigenvalue $\lambda$ for $A$ (with multiplicity 2).

(b) Find an eigenvector $v_1$ for $\lambda$ with integer components (to make things simple).

(c) Find a vector $v_2$, again with integer components, such that

$$(A - \lambda I_2)v_2 = v_1.$$

(d) Let $P$ be the matrix with columns $v_1$ and $v_2$ (in that order), and show that $P^{-1}AP$ is in Jordan form.

**Problem 2.** Find all possible Jordan forms for a matrix with a single real eigenvalue $u \in \mathbb{R}$ of multiplicity 4 (up to permutations of the blocks).

PROBLEM 3. Let $A \in M_4(\mathbb{R})$ with two *not necessarily distinct* real eigenvalues $u$ and $v$ and one pair of conjugate non-real eigenvalues $a \pm bi$. Find the possible real Jordan forms for $A$ (up to permutation of blocks and $\pm b$).

PROBLEM 4. Describe all possible Jordan forms for a real $2\times 2$ degenerate system (i.e., with determinant 0). There are two possibilities for which 0 is a repeated eigenvalue and an infinite class of possibilities for which 0 is an eigenvalue of multiplicity 1. For each, describe the solution to $x' = Jx$ with initial condition $x_0 = (\alpha, \beta) \in \mathbb{R}^2$.

PROBLEM 5. Solve the system

$$x' = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

with initial condition $x(0) = (7, 5) \in \mathbb{R}^2$, using the method from section 1.10 of our text. (Using the notation from section 1.10, take $\Phi(t) = e^{At}$. Show your work).