Math 322 Homework 4

PROBLEM 1. (Product rule). Let $M_{m\times n}(F)$ denote the space of $m \times n$ matrices over $F = \mathbb{R}$ or $\mathbb{C}$. The derivative of a function

$$\begin{aligned} F &\to M_{m\times n}(F) \\ t &\mapsto A(t) \end{aligned}$$

with respect to t is defined entrywise:

$$(A(t)')_{ij} = (A(t)_{ij})'.$$

Let $A(t)$ and $B(t)$ be two such functions into $M_{m\times p}(F)$ and $M_{p\times n}$, respectively. Use the ordinary product rule from one-variable calculus to prove the product rule

$$(A(t)B(t))' = A(t)'B(t) + A(t)B(t)'.$$

(Use the definition of multiplication of matrices using summation notation. Do not write out matrices with ellipses.)

PROBLEM 2. Consider the system

$$\begin{aligned} x_1' &= x_1 + 4x_2 \\ x_2' &= 4x_1 + x_2 \end{aligned}$$

Find the solution to this system with initial condition $x(0) = (1, 3)$ by diagonalizing a matrix and exponentiating by hand. Don't use a computer (except to check your work, if you'd like), and show your work.

PROBLEM 3. Let $A \in M_n(F)$ and let $W \subseteq F^n$ be a subspace. Suppose W is invariant under A, i.e., $Aw \in W$ for all $w \in W$. Let $x' = Ax$ have solution $x(t)$ with $x(0) = x_0 \in W$. The goal of this problem is to show that $x(t)$ never leaves the subspace W. To prove this, fix t and define the sequence

$$x_n = \left(\sum_{k=0}^{n} \frac{A^k t^k}{k!} \right) x_0$$

for each $n \geq 0$. Since $Ax_0 \in W$, it easily follows that $x_n \in W$ for all n.

Now, the space W is complete, i.e., every Cauchy sequence in W converges to a point in $w \in W$. That's because W is linearly isomorphic to F^m where $m = \dim W$, which is complete.) Therefore, if we can show that (x_n) is a Cauchy sequence, the result will follow since

$$x(t) = e^{At}x_0 = \lim_{n\to\infty} x_n = w \in W.$$

Problem. Your job is to prove that the sequence (x_n) is a Cauchy sequence. You may use the fact that e^{At} is Cauchy for each t (as shown in class). Give an ε-N proof. Lemma 1, from the lecture on Monday Week 3 may be of use.

PROBLEM 4. (D'Alembert reduction trick.) Suppose you have found a solution $a(t)$ to a differential equation of degree n

$$q_n(t)y^{(n)} + q_{n-1}(t)y^{(n-1)} + \cdots + q_1(t)y' + q_0(t)y = 0.$$

Substitute $y(t) = u(t)a(t)$. Then, after cancellation, substitute $v = u'$ to get an equation of the form

$$aq_n(t)v^{(n-1)} + p_{n-2}(t)v^{(n-2)} + \cdots + p_0(t)v = 0,$$

of degree $n - 1$.

(a) Write out the details of this reduction procedure for the case $n = 3$.

(b) Apply the reduction procedure to find the most general solution to

$$t^2y'' - 3ty' + 4y = 0$$

given the solution $a(t) = t^2$. You may assume that the initial conditions are such that $t > 0$ and the substituted quantities are positive.