Math 322

April 22, 2022

- Presentation date
- topic assignments

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Hamiltonian system with n degrees of freedom:

$$x' = (x_1', \dots, x_n') = H_y := \frac{\partial H}{\partial y} = \left(\frac{\partial H}{\partial y_1}, \dots, \frac{\partial H}{\partial y_n}\right)$$

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$H =$ Hamiltonian or total energy of the system.

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&= \frac{\partial H}{\partial x} \cdot \frac{\partial H}{\partial y} - \frac{\partial H}{\partial y} \cdot \frac{\partial H}{\partial x} \\
&= 0.
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Solutions lie on level sets for H.

Example

Let $H(x, y) = y \sin(x)$ and consider the Hamiltonian with one degree of freedom:

$$\begin{aligned} x' &= H_y = \sin(x) \\ y' &= -H_x = -y \cos(x). \end{aligned}$$

Critical points of a Hamiltonian system

$$x' = \frac{\partial H}{\partial y} = \left(\frac{\partial H}{\partial y_1}, \dots, \frac{\partial H}{\partial y_n}\right)$$

$$y' = -\frac{\partial H}{\partial x} = -\left(\frac{\partial H}{\partial x_1}, \dots, \frac{\partial H}{\partial x_n}\right)$$

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These occur where the graph of H

$$\mathrm{graph}(H) := \left\{(x, y, H(x, y)) \subset \mathbb{R}^{2n+1} : (x, y) \in E\right\},$$

has a horizontal tangent space.

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$$H(0) + \underbrace{\frac{1}{2}\frac{\partial^2 H}{\partial x_1^2}(0)x_1^2 + \frac{\partial^2 H}{\partial x_1 \partial x_2}(0)x_1 x_2 + \cdots + \frac{1}{2}\frac{\partial^2 H}{\partial y_n^2}(0)y_n^2}_{Q(x,y)}.$$

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Linear algebra (spectral theorem): after a linear change of coordinates, Q has the form

$$\widetilde{Q} = v_1^2 + \cdots + v_k^2 - v_{k+1}^2 - \cdots - v_r^2$$

Example, continued

$$x' = H_y = \sin(x)$$
$$y' = -H_x = -y\cos(x)$$

Lemma

Corollary. Let $p \in \mathbb{R}^{2n}$. Suppose there is a solution $\gamma(t) = (x(t), y(t))$ such that $\gamma(0) \neq p$ but such that $\gamma(t) \to p \in \mathbb{R}^{2n}$ as either $t \to \infty$ or $t \to -\infty$.

Lemma

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Then p is not a strict minimum or maximum of H.

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Theorem. Consider a Hamiltonian system with one degree of freedom and total energy function $H(x, y)$.

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Proof. Linearized system:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} H_{yx} & H_{yy} \\ -H_{xx} & -H_{xy} \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}.$$

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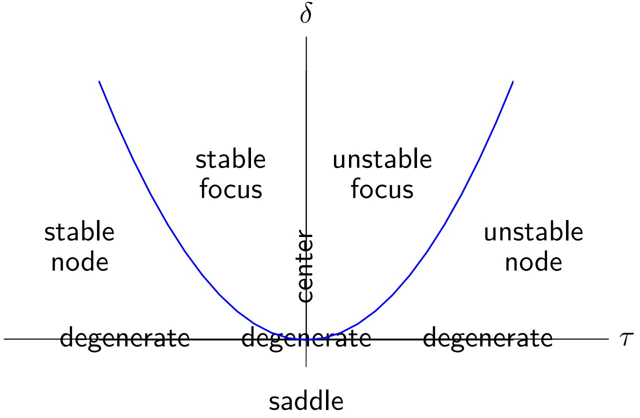

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Hamiltonian with $H_y = y$ and $H_x = -f(x)$. It follows that

$$H(x, y) = T(y) + U(x)$$

where $T(y) = \frac{1}{2}y^2$ (kinetic energy) and $U(x) = -\int_{x_0}^{x} f(s)\,ds$ (potential energy).

Newtonian system with one degree of freedom

Theorem. The critical points of this Newtonian system lie on the x-axis. The point $(x_0, 0)$ is a critical point iff x_0 is a critical point of the function $U(x)$, i.e., iff $U'(x_0) = 0$. Suppose that H is analytic. Then,

1. If x_0 is a strict local maximum for U, then $(x_0, 0)$ is a saddle for the system.
2. If x_0 is a strict local minimum for U, then $(x_0, 0)$ is a center for the system.
3. If x_0 is a horizontal inflection point for U (which means its first nonzero derivative at x_0 is of an odd order), then $(x_0, 0)$ is a cusp (i.e., two hyperbolic sectors and two separatrices).

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Potential energy: $U(x) = \int_0^x \sin(s)\, ds = 1 - \cos(x)$