Math 322

April 13, 2022

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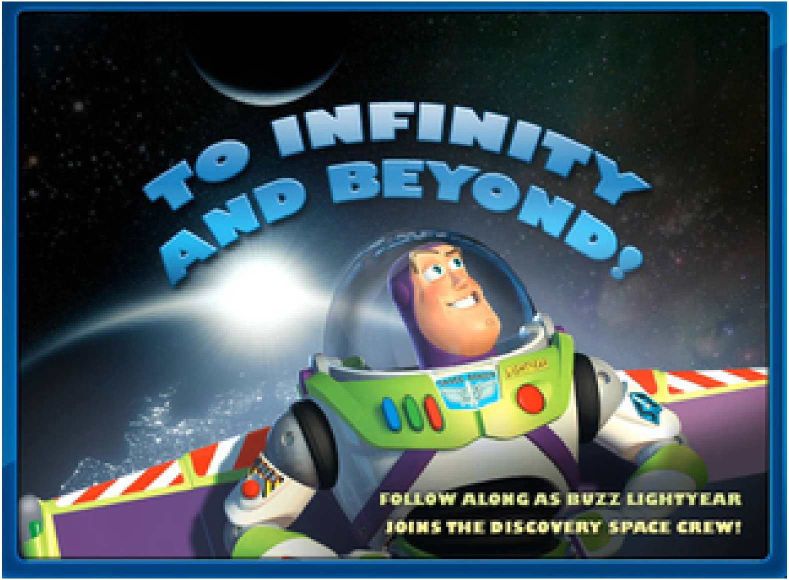
- Presentations

Critical points at infinity



Critical points at infinity

Consider a planar polynomial system:

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x' &= P(x, y) \\
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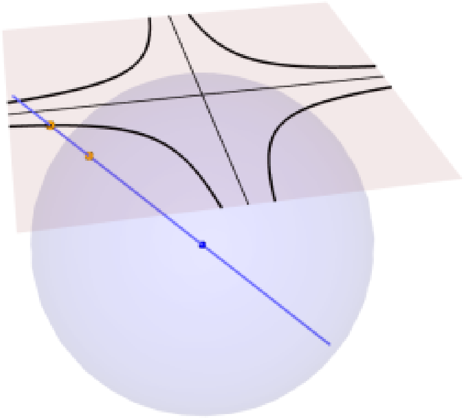
Our goal now is to look at critical points of this system "at infinity".

Critical points at infinity

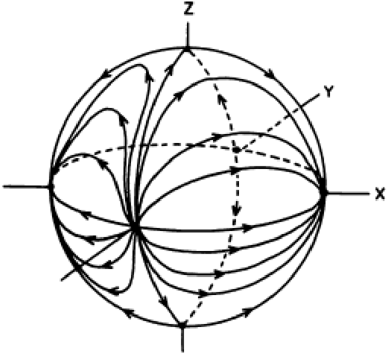
Induced flow on the sphere:



(Sage demo)

Critical points at infinity

Using central projection, project the flow from the plane $z = 1$ to the sphere:



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The corresponding point on the sphere is

$$(X, Y, Z) = \frac{1}{\sqrt{x^2 + y^2 + 1}}(x, y, 1).$$

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Clear denominators and regroup terms:

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Meaning: the solution curve $\gamma(t) = (X(t), Y(t), Z(t))$ has velocity normal to the vector $N := (QZ, -PZ, PY - QX)$:

$$N \cdot \gamma'(t) = 0.$$

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Let $d = \max\{\deg(P), \deg(Q)\}$ and scale to remove denominators:

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What happens as $Z \to 0$, i.e., as we approach the equator?

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Case 2: Otherwise, we look for critical points along the equator for which $P^* Y - Q^* X = 0$.

Analyzing critical points at infinity

1. Find points $(a, b, 0)$ satisfying $P^*Y - Q^*X = 0$.

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