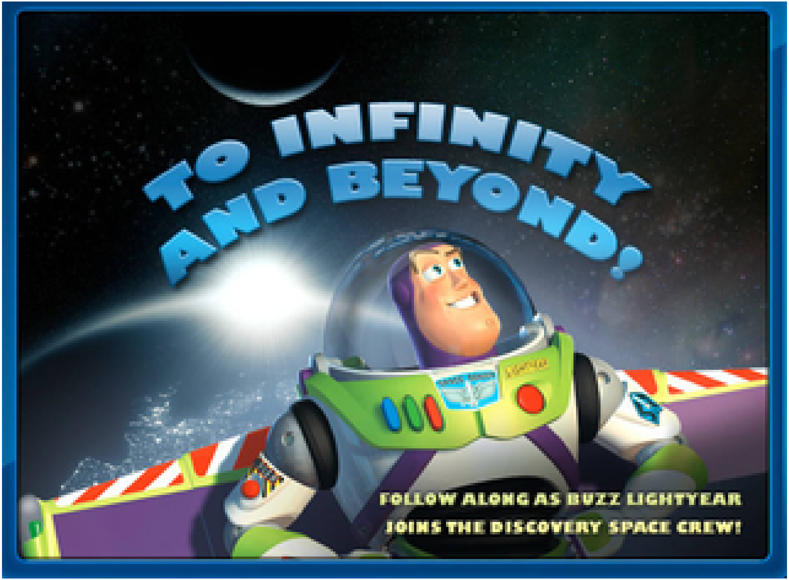
Math 322

April 15, 2022

Critical points at infinity



Critical points at infinity

Consider a planar polynomial system:

$$\begin{aligned} x' &= P(x, y) \\ y' &= Q(x, y) \end{aligned} \tag{1}$$

where P and Q are polynomials.

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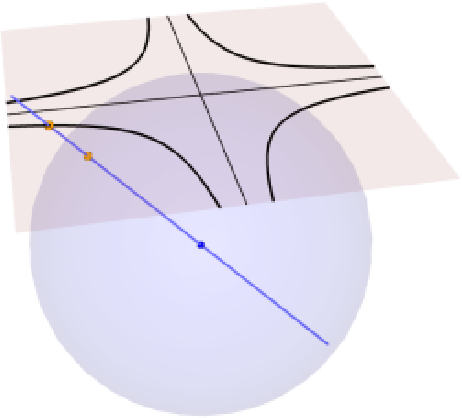
Our goal now is to look at critical points of this system "at infinity".

Critical points at infinity

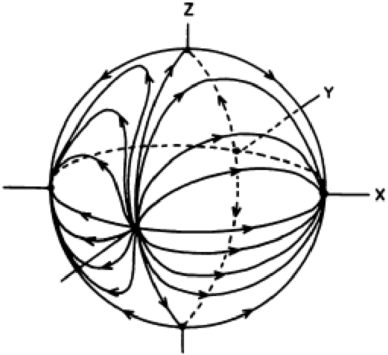
Induced flow on the sphere:



(Sage demo)

Critical points at infinity

Using central projection, project the flow from the plane $z = 1$ to the sphere:



Critical points at infinity

STEP 1. Clear denominators in the equation

$$yP\left(\frac{x}{z}, \frac{y}{z}\right) - xQ\left(\frac{x}{z}, \frac{y}{z}\right) = 0,$$

and then set $z = 0$.

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STEP 2. To analyze these, we will project the flow on the sphere to either the plane $x = 1$ or the plane $y = 1$.

Projection to $x = 1$ plane

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Projected system:

$$x' = P(x, y) \quad \Rightarrow \quad \left(\frac{1}{v}\right)' = P\left(\frac{1}{v}, \frac{u}{v}\right)$$

$$y' = Q(x, y) \quad \Rightarrow \quad \left(\frac{u}{v}\right)' = Q\left(\frac{1}{v}, \frac{u}{v}\right)$$

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Points of interest on equator $(z = 0)$ occur where where $x = 0$ or $y = 0$:

$$(1, 0, 0) \quad \text{and} \quad (0, 1, 0)$$

Example: check $(1, 0, 0)$

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So $(1, 0, 0)$ is a **source**.

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So $(0, 1, 0)$ is a **sink**.

Global phase portrait