# Math 322

April 6, 2022

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We will also require that $\gamma$ is *piece-wise* smooth.

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**Definition.** The index $I_f(C)$ of $C$ relative to $f$ is

$$I_f(C) := \frac{\Delta\theta}{2\pi}$$

where $\Delta\theta$ is the change in angle of $f(x, y)$ as $(x, y)$ travels around $C$ counterclockwise.

# Examples

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How would the index change if $C$ were replaced by an ellipse?

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Write $\gamma(t) = (x(t), y(t))$, and use polar coordinates:

$$\begin{aligned} f(\gamma(t)) &= (P(x(t), y(t)), Q(x(t), y(t))) \\ &= (r(t)\cos(\theta(t)), r(t)\sin(\theta(t))). \end{aligned}$$

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\end{aligned}$$

$$\longrightarrow \quad I_f(C) = \frac{\Delta\theta}{2\pi} = \frac{1}{2\pi} \oint_C \frac{P\,dQ - Q\,dP}{P^2 + Q^2} \quad \longleftarrow$$

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$$= \frac{1}{2\pi} \int_{t=0}^{2\pi} \, dt = 1.$$

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**Corollary.** If $C$ and $C'$ are Jordan curves containing the same finite set of critical points in their interiors, then $I_f(C) = I_f(C')$.

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**Definition.** Let $p$ be an isolated critical point of $f$. Define the *index of* $x$ *relative to* $f$ to be

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where $C$ is any Jordan curve containing $p$ as its only interior critical point. (This is well-defined from the previous corollary.)

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**Theorem.** Let $p_1, \ldots, p_n$ be the critical points inside $C$. Then

$$I_f(C) = \sum_{i=1}^{n} I_f(p_i).$$