Math 322

April 4, 2022

Grogu

Statistics job talk

Speaker: Chetkar Jha

Title: *Multiple Hypothesis Testing Approach to Estimate the Number of Networks in Sparse Stochastic Block Models*

4:45–5:35 Tuesday, Bio 19

Example

$$\begin{aligned} x' &= -2y + yz \\ y' &= x - xz \\ z' &= xy \end{aligned}$$

Example

$$x' = -2y + yz$$
$$y' = x - xz$$
$$z' = xy$$

eigenvalues $= 0, \pm\sqrt{2}i$ (origin is nonhyperbolic equilibrium point)

Example

$$x' = -2y + yz$$
$$y' = x - xz$$
$$z' = xy$$

eigenvalues $= 0, \pm\sqrt{2}i$ (origin is nonhyperbolic equilibrium point)

Look for Liapunov function of the form $V = ax^2 + by^2 + cz^2$:

Example

$$\begin{aligned} x' &= -2y + yz \\ y' &= x - xz \\ z' &= xy \end{aligned}$$

eigenvalues $= 0, \pm\sqrt{2}i$ (origin is nonhyperbolic equilibrium point)

Look for Liapunov function of the form $V = ax^2 + by^2 + cz^2$:

$$\dot{V} = 2axx' + 2byy' + 2czz'$$

Example

$$\begin{aligned} x' &= -2y + yz \\ y' &= x - xz \\ z' &= xy \end{aligned}$$

eigenvalues $= 0, \pm\sqrt{2}i$ (origin is nonhyperbolic equilibrium point)

Look for Liapunov function of the form $V = ax^2 + by^2 + cz^2$:

$$\begin{aligned} \dot{V} &= 2axx' + 2byy' + 2czz' \\ &= 2ax(-2y + yz) + 2by(x - xz) + 2cz(xy) \end{aligned}$$

Example

$$x' = -2y + yz$$
$$y' = x - xz$$
$$z' = xy$$

eigenvalues $= 0, \pm\sqrt{2}i$ (origin is nonhyperbolic equilibrium point)

Look for Liapunov function of the form $V = ax^2 + by^2 + cz^2$:

$$\begin{aligned}\dot{V} &= 2axx' + 2byy' + 2czz' \\ &= 2ax(-2y + yz) + 2by(x - xz) + 2cz(xy) \\ &= 2(-2a + b)xy + (a - b + c)xyz.\end{aligned}$$

Example

$$x' = -2y + yz$$
$$y' = x - xz$$
$$z' = xy$$

eigenvalues $= 0, \pm\sqrt{2}i$ (origin is nonhyperbolic equilibrium point)

Look for Liapunov function of the form $V = ax^2 + by^2 + cz^2$:

$$\begin{aligned} \dot{V} &= 2axx' + 2byy' + 2czz' \\ &= 2ax(-2y + yz) + 2by(x - xz) + 2cz(xy) \\ &= 2(-2a + b)xy + (a - b + c)xyz. \end{aligned}$$

Take $a = c = 1$ and $b = 2$.

Example

$$
\begin{aligned}
x' &= -2y + yz \\
y' &= x - xz \\
z' &= xy
\end{aligned}
$$

eigenvalues $= 0, \pm\sqrt{2}i$ (origin is nonhyperbolic equilibrium point)

Look for Liapunov function of the form $V = ax^2 + by^2 + cz^2$:

$$
\begin{aligned}
\dot{V} &= 2axx' + 2byy' + 2czz' \\
&= 2ax(-2y + yz) + 2by(x - xz) + 2cz(xy) \\
&= 2(-2a + b)xy + (a - b + c)xyz.
\end{aligned}
$$

Take $a = c = 1$ and $b = 2$. (Sage demo)

Equilibrium points for planar systems

Consider

$$\begin{aligned} x' &= P(x, y) \\ y' &= Q(x, y) \end{aligned}$$

having $(0, 0)$ as an equilibrium point.

Equilibrium points for planar systems

Consider

$$\begin{aligned} x' &= P(x, y) \\ y' &= Q(x, y) \end{aligned}$$

having $(0, 0)$ as an equilibrium point. Possibilities:

Equilibrium points for planar systems

Consider

$$x' = P(x, y)$$
$$y' = Q(x, y)$$

having $(0, 0)$ as an equilibrium point. Possibilities:

- The origin is a **center** if there exists $\delta > 0$ such that every trajectory with initial condition in $B_\delta \setminus \{(0, 0)\}$ is a closed curve containing $(0, 0)$ in its interior.

Equilibrium points for planar systems

Consider

$$x' = P(x, y)$$
$$y' = Q(x, y)$$

having $(0, 0)$ as an equilibrium point. Possibilities:

- The origin is a **center** if there exists $\delta > 0$ such that every trajectory with initial condition in $B_\delta \setminus \{(0, 0)\}$ is a closed curve containing $(0, 0)$ in its interior.
- Let $r(t, r_0, \theta_0)$ and $\theta(t, r_0, \theta_0)$ denote the solution to our system in polar coordinates and with initial conditions $r(0) = r_0$ and $\theta(0) = \theta_0$. The origin is a **stable focus** if there exists $\delta > 0$ such that $0 < r_0 < \delta$ and $\theta_0 \in \mathbb{R}$ imply $r(t, r_0, \theta_0) \to (0, 0)$ and $|\theta(t, r_0, \theta_0)| \to \infty$ as $t \to \infty$. It is an **unstable focus** if the same holds as $t \to -\infty$.

Equilibrium points for planar systems

- The origin is a **stable node** if there exists $\delta > 0$ such that for $0 < r_0 < \delta$ and $\theta_0 \in \mathbb{R}$, we have $r(t, r_0, \theta_0) \to (0,0)$ as $t \to \infty$ and $\lim_{t\to\infty} \theta(t, r_0, \theta_0)$ exists. In other words, the trajectories approach the origin with a well-defined tangent. It's an **unstable node** if the same holds with $t \to -\infty$. A node is called *proper* if every ray through the origin is tangent to some trajectory.

Equilibrium points for planar systems

- The origin is a **stable node** if there exists $\delta > 0$ such that for $0 < r_0 < \delta$ and $\theta_0 \in \mathbb{R}$, we have $r(t, r_0, \theta_0) \to (0, 0)$ as $t \to \infty$ and $\lim_{t\to\infty} \theta(t, r_0, \theta_0)$ exists. In other words, the trajectories approach the origin with a well-defined tangent. It's an **unstable node** if the same holds with $t \to -\infty$. A node is called *proper* if every ray through the origin is tangent to some trajectory.
- The origin is a **topological saddle** if it is locally homeomorphic to a saddle for a linear system.

Equilibrium points for planar systems

- The origin is a **stable node** if there exists $\delta > 0$ such that for $0 < r_0 < \delta$ and $\theta_0 \in \mathbb{R}$, we have $r(t, r_0, \theta_0) \to (0, 0)$ as $t \to \infty$ and $\lim_{t\to\infty} \theta(t, r_0, \theta_0)$ exists. In other words, the trajectories approach the origin with a well-defined tangent. It's an **unstable node** if the same holds with $t \to -\infty$. A node is called *proper* if every ray through the origin is tangent to some trajectory.
- The origin is a **topological saddle** if it is locally homeomorphic to a saddle for a linear system.
- The origin is a **center-focus** if there exists a sequence of closed solution curves Γ_n with Γ_{n+1} in the interior of Γ_n such that $\Gamma_k \to (0, 0)$ as $k \to \infty$ and such that every solution with initial condition between Γ_n and Γ_{n+1} spirals toward either Γ_n or Γ_{n+1} as $t \to \pm\infty$.

Example of a center focus

$$x' = -y + x\sqrt{x^2 + y^2}\sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right)$$

$$y' = x + y\sqrt{x^2 + y^2}\sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right).$$

Example of a center focus

$$x' = -y + x\sqrt{x^2+y^2}\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$

$$y' = x + y\sqrt{x^2+y^2}\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right).$$

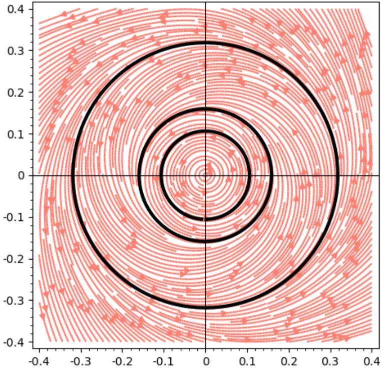
In polar coordinates:

$$r' = r^2\sin\left(\frac{1}{r}\right)$$

$$\theta' = 1$$

for $r > 0$, and $r' = 0$ for $r = 0$.

Example of a center focus



Comparison with linearized system: hyperbolic case

LINEARIZED	NONLINEAR
saddle	saddle

Comparison with linearized system: hyperbolic case

LINEARIZED	NONLINEAR
saddle	saddle
center	center, focus, center-focus

Comparison with linearized system: hyperbolic case

LINEARIZED	NONLINEAR
saddle	saddle
center	center, focus, center-focus
center	center, focus in analytic case

Comparison with linearized system: hyperbolic case

LINEARIZED	NONLINEAR
saddle	saddle
center	center, focus, center-focus
center	center, focus in analytic case
node	node or focus

Comparison with linearized system: hyperbolic case

LINEARIZED	NONLINEAR
saddle	saddle
center	center, focus, center-focus
center	center, focus in analytic case
node	node or focus
node	node in C^2 case

Comparison with linearized system: hyperbolic case

LINEARIZED	NONLINEAR
saddle	saddle
center	center, focus, center-focus
center	center, focus in analytic case
node	node or focus
node	node in C^2 case
focus	node or focus

Comparison with linearized system: hyperbolic case

LINEARIZED	NONLINEAR
saddle	saddle
center	center, focus, center-focus
center	center, focus in analytic case
node	node or focus
node	node in C^2 case
focus	node or focus
focus	focus in C^2 case

Comparison with linearized system: hyperbolic case

LINEARIZED	NONLINEAR
saddle	saddle
center	center, focus, center-focus
center	center, focus in analytic case
node	node or focus
node	node in C^2 case
focus	node or focus
focus	focus in C^2 case

See course homepage for Perron's example of a node that turns into a focus upon the addition of non-linear terms:

$$x' = -x - \frac{y}{\log\sqrt{x^2+y^2}}$$
$$y' = -y + \frac{x}{\log\sqrt{x^2+y^2}}$$

Comparison with linearized system: non-hyperbolic equilibria

Suppose the linearized system at x_0 is nonzero, and x_0 is non-hyperbolic. The only new possibilities are (see Perko):

Comparison with linearized system: non-hyperbolic equilibria

Suppose the linearized system at x_0 is nonzero, and x_0 is non-hyperbolic. The only new possibilities are (see Perko):

- **saddle-nodes** (two hyperbolic sectors, one parabolic sector)

Comparison with linearized system: non-hyperbolic equilibria

Suppose the linearized system at x_0 is nonzero, and x_0 is non-hyperbolic. The only new possibilities are (see Perko):

- **saddle-nodes** (two hyperbolic sectors, one parabolic sector)
- **critical points with elliptic domains** (one elliptic sector, one hyperbolic sector, two parabolic sectors, four separatrices)

Comparison with linearized system: non-hyperbolic equilibria

Suppose the linearized system at x_0 is nonzero, and x_0 is non-hyperbolic. The only new possibilities are (see Perko):

- **saddle-nodes** (two hyperbolic sectors, one parabolic sector)
- **critical points with elliptic domains** (one elliptic sector, one hyperbolic sector, two parabolic sectors, four separatrices)
- **cusps** (two hyperbolic sectors, two separatrices):