Math 322

March 28, 2022

Stability of equilibrium point

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3. In any case, if an equilibrium point x_0 is stable, then no eigenvalue of Df_{x_0} has positive real part (even in the non-hyperbolic case).

Liapunov functions

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Proof ...

Liapunov function example

Consider

$$x' = -y^3$$
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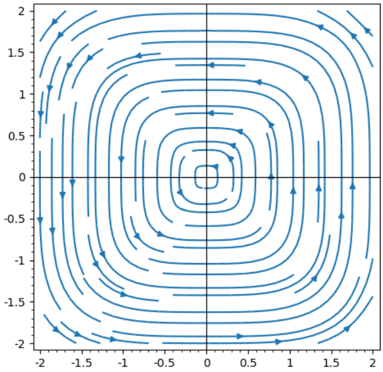
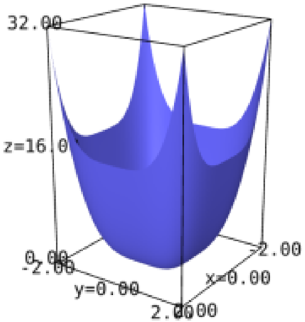
Check that trajectories sit on level sets for V.

Example, continued





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Theorem. Let $f \in C^1(E)$ and $f(x_0) = 0$. Let $V\colon E \to \mathbb{R}$ also be C^1 (continuously differentiable). Suppose that $V(p) \geq 0$ and $V(p) = 0$ if and only if $p = x_0$.

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2. If $\dot{V}$ is negative definite ($\dot{V}(p) < 0$ for all $p \in E \setminus \{x_0\}$) then x_0 is asymptotically stable.

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3. If $\dot{V}$ is positive definite ($\dot{V}(p) > 0$ for all $p \in E \setminus \{x_0\}$), then x_0 is unstable.