Math 322

April 1, 2022

Statistics search job talk

Eli Wolff, University of Oregon

Two-Dimensional Electrostatics and Universality in Random Matrix Theory

4:45–5:35 Thursday, E314

Projective space

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$$\{\text{conics}\} = \mathbb{P}^5$$

Stability of equilibrium point

Definition. An equilibrium point x_0 for a system $x' = f(x)$ is *stable* if for each open neighborhood U of x_0, there exists another open neighborhood W of x_0 such that if $p \in W$, then $\phi(t, p) \in U$ for all $t \geq 0$.

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We say x_0 is *asymptotically stable* if it has an open neighborhood W such that $\lim_{t \to \infty} \phi_t(p) = x_0$ for all $p \in W$.

Liapunov functions

Theorem. Let $f \in C^1(E)$ and $f(x_0) = 0$. Let $V\colon E \to \mathbb{R}$ also be C^1 (continuously differentiable). Suppose that $V(p) \geq 0$ and $V(p) = 0$ if and only if $p = x_0$.

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1. If $\dot{V}$ is negative semidefinite ($\dot{V}(p) \leq 0$ for all $p \in E \setminus \{x_0\}$) then x_0 is stable.

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3. If $\dot{V}$ is positive definite ($\dot{V}(p) > 0$ for all $p \in E \setminus \{x_0\}$), then x_0 is unstable.

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\begin{aligned}
x' &= -2y + yz \\
y' &= x - xz \\
z' &= xy
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